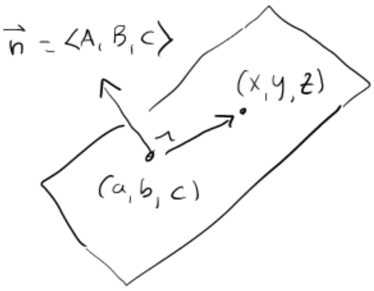
Section 14.6 Tangent Planes and Differentials

Recall Equation of plane normal to $\vec{n} = \langle A, B, C\rangle$ and containing point (a, b, c) is:

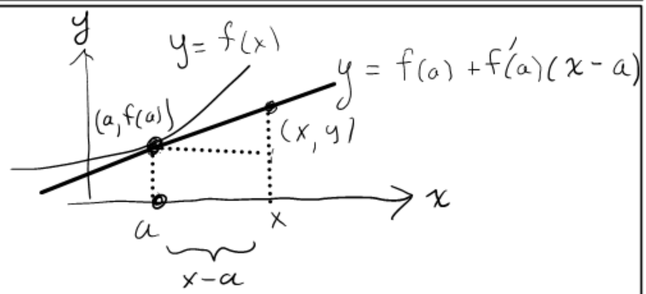
- $\langle A, B, C\rangle \cdot \langle x-a,\ y-b,\ z-c\rangle = 0$
- $A(x-a) + B(y-b) + C(z-c) = 0$
- $Ax + By + Cz = Aa + Bb + Cc$.



Recall Line tangent to $y = f(x)$ at $x = a$ has equation

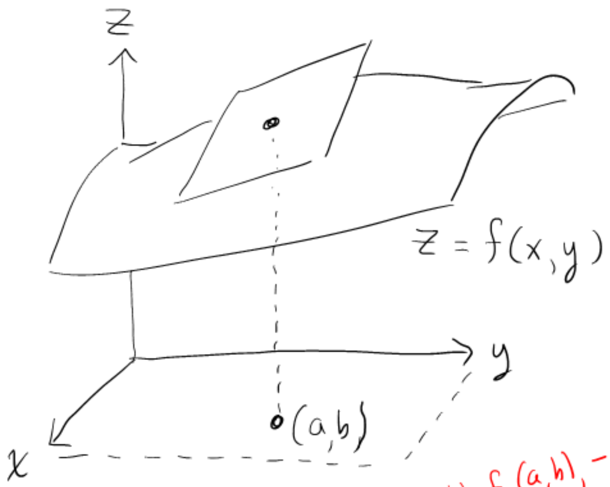
$$f'(a) = \frac{y - f(a)}{x - a} = \frac{\text{rise}}{\text{run}}$$

$$\rightsquigarrow \quad y = f(a) + f'(a)(x-a)$$



Tangent line is an approximation of $y = f(x)$ at $x = a$ by a linear function

The graph of a function $z = f(x, y)$ has a tangent plane at $(x, y) = (a, b)$, touching the graph at the point $(a, b, f(a, b))$. This plane is an approximation to $z = f(x, y)$ at (a, b) by a "linear function" of two variables



Question

What is the equation of this plane?

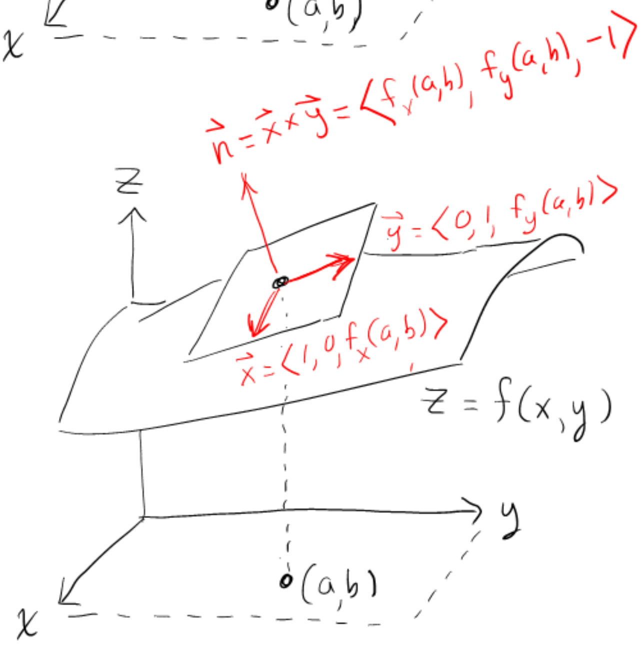
Answer (From picture on right)

Equation of plane tangent to graph of $z = f(x, y)$ at point $(a, b, f(a, b))$ is

$$f_x(a,b)(x-a) + f_y(a,b)(y-a) - (z - f(a,b)) = 0$$

or $Z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

The normal vector is $\langle -f_x(a,b), -f_y(a,b), 1\rangle$



Example Find equation of tangent plane to $z = f(x,y) = x^2 + y^2$ at point $(2, 5, 29)$.

$$\left.\begin{aligned} f_x(x,y) &= 2x \\ f_y(x,y) &= 2y \end{aligned}\right\}$$

$$Z = f(2,5) + f_x(2,5)(x-2) + f_y(2,5)(y-5)$$

$$Z = 29 + 4(x-2) + 10(y-5)$$

$$Z = 4x + 10y - 29$$

Linearization

The *linearization* of $f(x,y)$ at (a,b) is

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

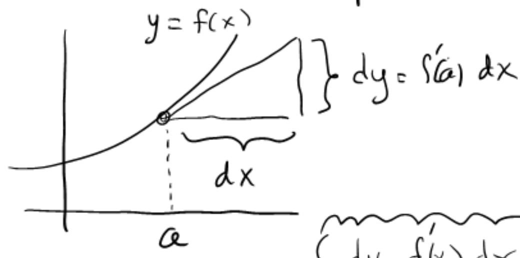
That is, it is the approximation to $f(x,y)$ near (a,b) by the tangent plane. Then $f(x,y) \approx L(x,y)$ for (x,y) near (a,b). This can be useful for estimating changes in $f(x,y)$ since $L(x,y)$ is usually much simpler than $f(x,y)$. Read the material in the text.

Do not need to know about "Error in the standard linear approximation"

Differentials

Recall that for $y=f(x)$, the *differentials* are variables dx and dy related by

$$\boxed{dy = f'(a)\,dx}$$



$dy = f'(x)\,dx \iff \dfrac{dy}{dx} = f'(x)$

Interpretation:

$dx = \Delta x$ = change in x

$dy = f'(a)dx \approx$ corresponding change in $f(x)$

For two variables, this plays out as follows

$dx = \Delta x$ = change in x

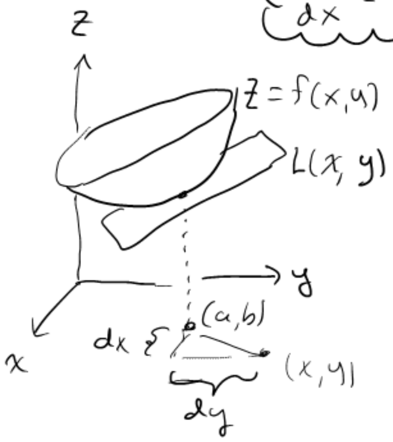
$dy = \Delta y$ = change in y.

called the total differential

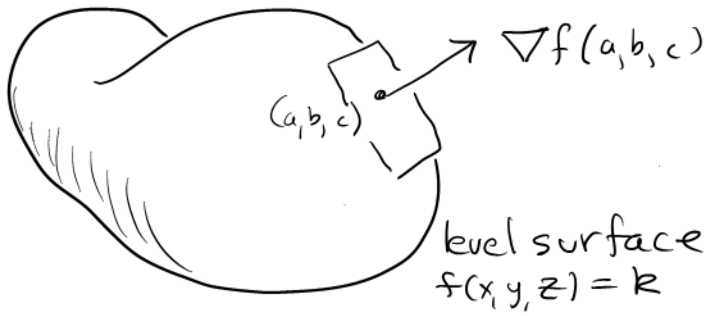
$$\boxed{dz = f_x(a,b)\,dx + f_y(a,b)\,dy}$$

$= f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$\approx$ approx change in $f(x,y)$ at (a,b) when a incremented by dx and b by dy.



Text also makes the point that given a function $f(x,y,z)$ and point (a,b,c) on a level surface $f(x,y,z)=k$, the normal to the surface at that point is $\nabla f(a,b,c)$.



Therefore equation of tangent plane at (a,b,c) is

$\nabla f(a,b,c)\cdot\langle x-a,\ y-b,\ z-c\rangle = 0$, which is

$$\boxed{f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c) = 0}$$

Read the examples in the text.