Section 14.4 The Chain Rule

Today we are concerned with finding partial derivatives of compositions.

Typical Example: $z = f(x, y) = f(g(u,v), h(u,v))$ $\quad \left\{ \begin{array}{l} \frac{\partial f}{\partial u} = ? \\ \frac{\partial f}{\partial v} = ? \end{array} \right.$

$x = g(u,v) \qquad y = h(u,v)$

In the one-variable case, the chain rule gives the answer

Chain Rule

If $y = f(x)$ and $x = g(u)$ [i.e. $y = f(g(u))$], then

$$\frac{dy}{du} = \frac{dy}{dx}\frac{dx}{du} = f'(x)\,g'(u) = f'(g(u))\,g'(u)$$

Today we examine analogous rules for multiple variables. But there are various ways this can play out. We'll start with the simplest case

Suppose $z = f(x, y) = f(g(t), h(t))$ ← Function of t, $\frac{dz}{dt} = ?$

$x = g(t) \qquad y = h(t)$

Ex $f(x,y) = y\cos x + 3$, $\quad g(t) = t^3$, $\quad h(t) = \tan t$

$f(g(t), h(t)) = \tan t \cos t^3 + 3$ ← Function of t

Theorem Suppose $w = f(x, y)$ and $x = g(t)$ and $h = y(t)$. so $w = f(x(t), y(t))$. Then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$
$$= f_x(x,y)\,g'(t) + f_y(x,y)\,h'(t)$$
$$= f_x(g(t), h(t))\,g'(t) + f_y(g(t), h(t))\,h'(t)$$

Note In stating this and other formulas from this section we assume all functions are differentiable

Example

Find $\frac{dw}{dt}$: $w = f(x,y) = y\cos x + 3$ $\begin{cases} x = t^3 \\ y = \tan t \end{cases}$

Method A: $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$

$$= -y\sin(x)\,3t^2 + \cos(x)\sec^2 t$$

$$= -\tan t\sin(t^3)\,3t^2 + \cos(t^3)\sec^2 t$$

Method B $z = y\cos x + 3 = \tan t\cos(t^3) + 3$

$$\frac{dz}{dt} = \sec^2 t\cos(t^3) + \tan t(-\sin(t^3)\,3t^2)$$

Just use the familiar chain rule - get same answer.
But that's not always the easiest approach.

Theorem

Suppose $z = f(x,y)$ $\begin{cases} x = g(r,s) \\ y = h(r,s) \end{cases}$, i.e $z = f(g(r,s), h(r,s))$

function of two variables r and s

Then: $$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

Example $w = f(x,y) = xy + x$ $\begin{cases} x = \sin(rs) \\ y = e^{r+s} \end{cases}$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} = (y+1)\cos(rs)s + x e^{r+s}$$

$$= (e^{r+s}+1)\cos(rs)s + \sin(rs)e^{r+s}$$

Alternate Method $w = f(\sin(rs), e^{r+s}) = \sin(rs)e^{r+s} + \sin(rs)$

$$\frac{\partial w}{\partial r} = \cos(rs)s\,e^{r+s} + \sin(rs)\,e^{r+s} + \cos(rs)s$$

= (same answer as above)

Overall View: Suppose

$$w = f(x_1, x_2, x_3, \dots, x_m) \text{ and } \begin{cases} x_1 = g_1(r_1, r_2, \dots, r_n) \\ x_2 = g_2(r_1, r_2, \dots, r_n) \\ \vdots \\ x_m = g_m(r_1, r_2, \dots, r_n) \end{cases}$$

Then:

$$\frac{\partial w}{\partial r_1} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial r_1} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial r_1} + \cdots + \frac{\partial w}{\partial x_m}\frac{\partial x_m}{\partial r_1}$$

$$\frac{\partial w}{\partial r_2} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial r_2} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial r_2} + \cdots + \frac{\partial w}{\partial x_m}\frac{\partial x_m}{\partial r_2}$$

$$\vdots$$

$$\frac{\partial w}{\partial r_n} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial r_n} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial r_n} + \cdots + \frac{\partial w}{\partial x_m}\frac{\partial x_m}{\partial r_n}$$

Example $w = f(x, y, z) = z + y\sin(x)$ $\begin{cases} x = r^2 s^3 \\ y = r + 5s \\ z = r\sin(s) \end{cases}$

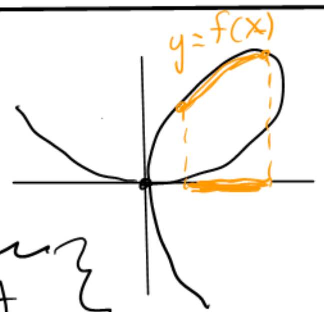
$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$= y\cos(x)\, 2rs^3 + \sin(x)\cdot 1 + 1\cdot\sin(s)$$

$$= (r+5s)\cos(r^2s^3)\, 2rs^3 + \sin(r^2s^3) + \sin(s)$$

Implicit Functions Revisited

Consider $x^3 + y^3 - 12xy = 0$ $\leftarrow$ y is not a function of x, but y is an implicit function of x



Implicit differentiation:

$$\frac{d}{dx}\left[x^3 + y^3 - 12xy\right] = \frac{d}{dx}[0] \rightsquigarrow \frac{dy}{dx} = -\frac{3x^2 - 12y}{3y^2 - 12x}$$

ALTERNATE METHOD Let $F(x, y) = x^3 + y^3 - 12xy = 0$

$x = x \quad y = f(x)$

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} = 0$$

$$(3x^2 - 12y)\cdot 1 + (3y^2 - 12x)\frac{dy}{dx} \rightsquigarrow \frac{dy}{dx} = -\frac{3x^2 - 12y}{3y^2 - 12x}$$