Today we are concerned with finding partial derivatives of compositions.

Typical Example: \[ z = f(x, y) = f(g(u, v), h(u, v)) \]

\[ x = g(u, v) \quad y = h(u, v) \]

In the one-variable case, the chain rule gives the answer:

Chain Rule

If \( y = f(x) \) and \( x = g(u) \) [i.e. \( y = f(g(u)) \)], then

\[ \frac{dy}{du} = \frac{dy}{dx} \frac{dx}{du} = f'(x) g'(u) = f'(g(u)) g'(u) \]

Today we examine analogous rules for multiple variables. But there are various ways this can play out. We'll start with the simplest case.

Suppose \( z = f(x, y) = f(g(t), h(t)) \)

\[ x = g(t) \quad y = h(t) \]

Example: \( f(x, y) = y \cos x + 3 \), \( g(t) = t^3 \), \( h(t) = \tan t \)

\[ f(g(t), h(t)) = \tan t \cos t^3 + 3 \]

Theorem: Suppose \( w = f(x, y) \) and \( x = g(t) \) and \( h = y(t) \).

so \( w = f(x(t), y(t)) \). Then

\[ \frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} \]

\[ = f_x(x(t), y(t)) g'(t) + f_y(x(t), y(t)) h'(t) \]

\[ = f(g(t), h(t)) g'(t) + f_y(g(t), h(t)) h'(t) \]

Note: In stating this and other formulas from this section, we assume all functions are differentiable.
Example
Find \( \frac{dw}{dt} \):
\[
 w = f(x,y) = y \cos x + 3 \quad \left\{ \begin{array}{l}
x = x^3 \\
y = \tan t
\end{array} \right.
\]

**Method A**
\[
\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}
\]
\[
= -y \sin(x) 3x^2 + \cos(x) \sec^2 t
\]
\[
= -\tan t \sin(x^3) 3x^2 + \cos(x^3) \sec^2 t
\]

**Method B**
\[
z = y \cos x + 3 = \tan t \cos(x^3) + 3
\]
\[
\frac{dz}{dt} = \sec^2 t \cos(x^3) + \tan t (-\sin(x^3) 3x^2)
\]

*(Just use the familiar chain rule - get same answer. But that's not always the easiest approach.)*

**Theorem**
Suppose \( z = f(x,y) \) \( \left\{ \begin{array}{l}x = g(r,s) \\
y = h(r,s)
\end{array} \right. \), i.e. \( z = f(g(r,s), h(r,s)) \)

Then:
\[
\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r}
\]
\[
= \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r}
\]

Example \( w = f(x,y) = xy + x \) \( \left\{ \begin{array}{l}x = \sin(rs) \\
y = e^{r+s}
\end{array} \right. \)

\[
\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r}
\]
\[
= (y+1) \cos(rs) s + x e^{r+s}
\]
\[
= (e^{r+s+1}) \cos(rs) s + \sin(rs) e^{r+s}
\]

**Alternate Method**
\( w = f(\sin(rs), e^{r+s}) = \sin(rs) e^{r+s} \)
\[
\frac{\partial w}{\partial r} = \cos(rs) s e^{r+s} + \sin(rs) e^{r+s} + \cos(rs) s
\]
\[
= (\text{same answer as above})
\]
Overall View: Suppose
\[ w = f(x_1, x_2, x_3, \ldots, x_m) \] and
\[ \{ x_1 = g_1(r_1, r_2, \ldots, r_n) \]
\[ x_2 = g_2(r_1, r_2, \ldots, r_n) \]
\[ \vdots \]
\[ x_m = g_m(r_1, r_2, \ldots, r_n) \]
Then:
\[ \frac{\partial w}{\partial r_1} = \frac{\partial w}{\partial x_1} \frac{\partial x_1}{\partial r_1} + \frac{\partial w}{\partial x_2} \frac{\partial x_2}{\partial r_1} + \ldots + \frac{\partial w}{\partial x_m} \frac{\partial x_m}{\partial r_1} \]
\[ \frac{\partial w}{\partial r_2} = \frac{\partial w}{\partial x_1} \frac{\partial x_1}{\partial r_2} + \frac{\partial w}{\partial x_2} \frac{\partial x_2}{\partial r_2} + \ldots + \frac{\partial w}{\partial x_m} \frac{\partial x_m}{\partial r_2} \]
\[ \vdots \]
\[ \frac{\partial w}{\partial r_n} = \frac{\partial w}{\partial x_1} \frac{\partial x_1}{\partial r_n} + \frac{\partial w}{\partial x_2} \frac{\partial x_2}{\partial r_n} + \ldots + \frac{\partial w}{\partial x_m} \frac{\partial x_m}{\partial r_n} \]

Example: \( w = f(x, y, z) = z + y \sin(x) \)
\[ \{ x = r^2 s^3 \]
\[ y = r + 5 s \]
\[ z = r \sin(s) \]
\[ \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r} \]
\[ = y \cos(x) z r s^3 + \sin(x) \cdot 1 + 1 \cdot \sin(s) \]
\[ = (r + 5 s) \cos(r^2 s^3) z r s^3 + \sin(r^2 s^3) + \sin(s) \]

Implicit Functions Revisited
Consider \( x^3 + y^3 - 12xy = 0 \)
Implicit differentiation:
\[ \frac{d}{dx} [x^3 + y^3 - 12xy] = \frac{d}{dx} [0] \]
\[ \frac{dy}{dx} = -\frac{3x^2 - 12y}{3y^2 - 12x} \]
Alternate Method: Let \( F(x, y) = x^3 + y^3 - 12xy = 0 \)
\[ \frac{\partial F}{\partial x} = \frac{\partial F}{\partial x} \frac{\partial x}{\partial x} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial x} = 0 \]
\[ (3x^2 - 12y) \frac{dy}{dx} + (3y^2 - 12x) \frac{dy}{dx} = 0 \]
\[ \frac{dy}{dx} = -\frac{3x^2 - 12y}{3y^2 - 12x} \]