Section 14.3 Partial Derivatives

Goal: Develop a notion of the derivative of a function $f(x,y)$

Notation Fixed point on the xy plane $\begin{cases} (x_0, y_0) \leftarrow \text{Text} \\ (a, b) \leftarrow \text{Me} \end{cases}$

Consider $z = f(x,y)$

Let a & b be constants.

$z = f(x,b) \leftarrow$ function of x

$z = f(a,y) \leftarrow$ function of y

Example

$f(x,y) = x^2\sqrt{y}$

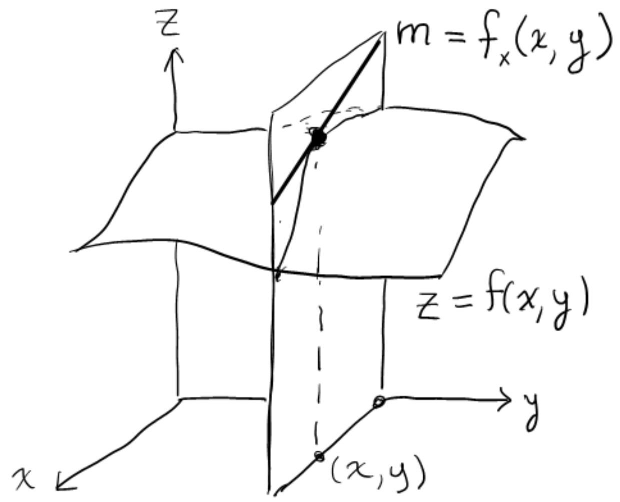
$\left.\begin{array}{l} f(x,b) = x^2\sqrt{b} \\ f(a,y) = a\sqrt{y} \end{array}\right\}$ can differentiate these

Derivative of $f(x,b)$:

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,b) - f(x,b)}{h}$$

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$\frac{\partial f}{\partial x} = f_x(x,y)$ is the derivative of $f(x,y)$ when y is held const. Called the partial derivative of $f(x,y)$ with respect to x.

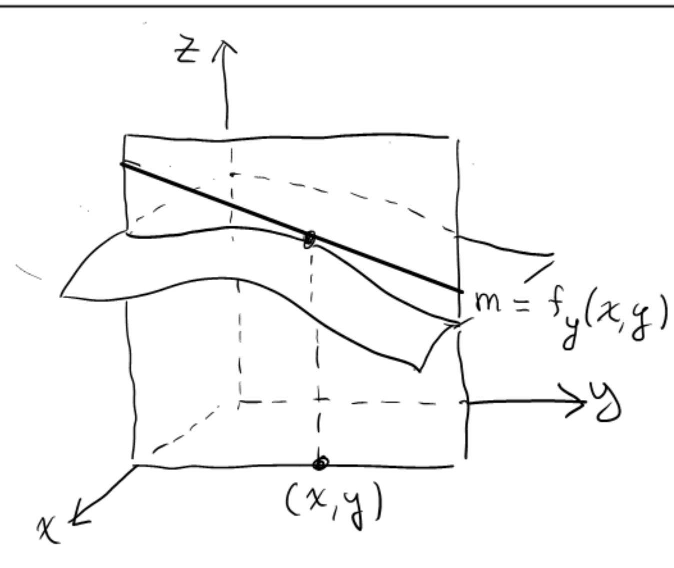


Derivative of $f(a,y)$

$$\frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(a,y+h) - f(a,y)}{h}$$

$$\frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

$\frac{\partial f}{\partial y} = f_y(x,y)$ is derivative of $f(x,y)$ with x held constant. Called the partial derivative of $f(x,y)$ with respect to y



Example $f(x y) = x^2 y^3 + x$

$$\frac{\partial f}{\partial x} = f_x(x,y) = 2xy^3 + 1$$

$$\frac{\partial f}{\partial y} = x^2 3y^2 + 0$$

Example $g(x,y) = x^2 \sin(xy)$

$$\frac{\partial f}{\partial x} = f_x(x,y) = 2x\sin(xy) + x^2\cos(xy)\,y$$

$$\frac{\partial f}{\partial y} = f_y(x,y) = x^2\cos(xy)x = x^3\cos(xy)$$

Notation

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= f_x(x,y) = \frac{\partial f}{\partial x}(x,y) = f_x = z_x = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[f] \\ \frac{\partial f}{\partial y} &= f_y(x,y) = \frac{\partial f}{\partial y}(x,y) = f_y = z_y = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[f] \end{aligned}}$$

Higher Partial Derivatives

$$f_{xx} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial^2 f}{\partial x^2} \qquad f_{yy} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial^2 f}{\partial y^2}$$

$$f_{xy} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial^2 f}{\partial y\,\partial x} \qquad f_{yx} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial^2 f}{\partial x\,\partial y}$$

Example $f(x,y) = x^3 e^{2y}$ $\begin{cases} f_x(x,y) = 3x^2 e^{2y} \\ f_y(x,y) = 2x^3 e^{2y} \end{cases}$

$$f_{xx}(x,y) = 6xe^{2y}$$

$$f_{yy}(x,y) = 4x^3 e^{2y}$$

$$\left.\begin{aligned} f_{xy}(x,y) &= \frac{\partial}{\partial y} f_x(x,y) = 6x^2 e^{2y} \\ f_{yx}(x,y) &= \frac{\partial}{\partial x} f_y(x,y) = 6x^2 e^{2y} \end{aligned}\right\}$$

These are the same – not a coincidence.

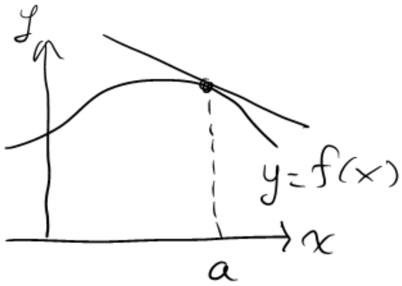
Theorem (Mixed Partial Theorem)

If f, f_x, f_y, f_{xy} and f_{yx} are all continuous in an open region containing (a,b), then

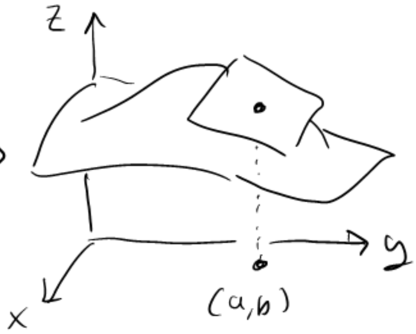
$$f_{xy}(a,b) = f_{yx}(a,b)$$

Differentiability

Recall that $y=f(x)$ differentiable at $x=a$ means $y=f(x)$ has tangent (non-vertical) at a.



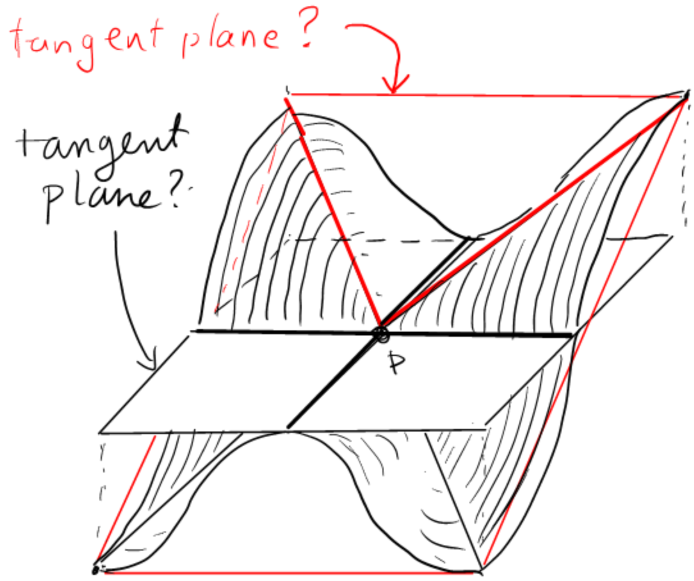
Similiarly, $y=f(x,y)$ being differentiable at (a,b) means that the graph has a tangent plane at (a,b), as shown $\rightarrow$



But this concept is somewhat subtle.

For example, here is a surface that seems to have two tangent planes at the point P.

We would not want to say that this function is differentiable at that point.



The text gives a somewhat technical definition of what it means for $f(x,y)$ to be differentiable at a point (a,b). The upshot of this definition is that $f(x,y)$ is differentiable if its graph has a unique tangent plane at (a,b). In other words, close up, the graph looks like a plane. We will have more to say about this later, but for now, one consequence.

Theorem $\left(\begin{array}{c} f_x \text{ and } f_y \text{ are continuous} \\ \text{on an open region } R \end{array}\right) \Longrightarrow \left(\begin{array}{c} f(x,y) \text{ is differentiable} \\ \text{on the region } R \end{array}\right)$

Theorem $\left(\begin{array}{c} f(x,y) \text{ is differentiable} \\ \text{at the point } (a,b) \end{array}\right) \Longrightarrow \left(\begin{array}{c} f(x,y) \text{ is continuous} \\ \text{at } (a,b) \end{array}\right)$