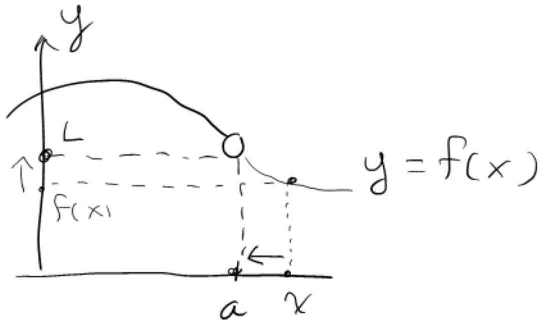
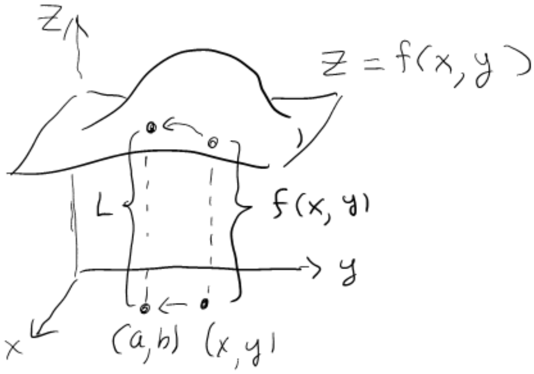
Section 14.2 Limits and Continuity

Given $y = f(x)$, we have an intuitive sense of what $\lim_{x \to a} f(x) = L$ means.
"$f(x)$ can be made arbitrarily close to L by choosing x sufficiently close to a."

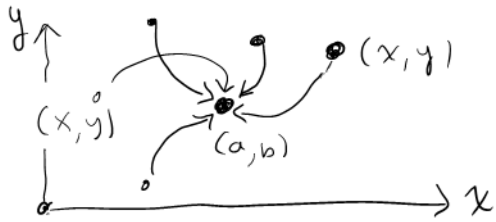


Similarly, $\lim_{(x,y) \to (a,b)} f(x,y) = L$

means $f(x,y)$ can be made arbitrarily close to L by choosing (x,y) sufficiently close to (a,b).



But this is a bit vague. For one thing, there are lots of ways for (x,y) to approach (a,b)



Also, what does "close" mean?
<u>Answer</u>: within some small distance ε or δ.

$\big(f(x,y)$ close to $L\big) \iff \big(f(x,y)$ is within ε units of $L\big) \iff |f(x,y) - L| < \varepsilon$

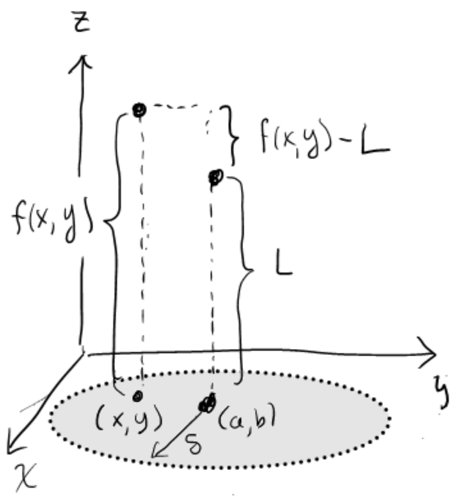
$\big((x,y)$ close to $(a,b)\big) \iff \big((x,y)$ is within a radius of δ from $(a,b)\big) \iff \sqrt{(x-a)^2 + (y-b)^2} < \delta$

Precise Definition

$\lim_{(x,y) \to (a,b)} f(x,y) = L$ means that

for any $\varepsilon > 0$ (no matter how small) there is a $\delta > 0$ (depending on ε) for which

$|f(x,y) - L| < \varepsilon$ whenever $\sqrt{(x-a)^2 + (y-b)^2} < \delta$



i.e. can make $f(x,y)$ this close to L ...

... by making (x,y) this close to (a,b)

Using this definition, the usual limit rules can be proved in this more general setting.

Ex $\displaystyle\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{\displaystyle\lim_{(x,y)\to(a,b)} f(x,y)}{\displaystyle\lim_{(x,y)\to(a,b)} g(x,y)}$ (provided both limits exist)

Read the complete list in the book — it should look familiar

Ex $\displaystyle\lim_{(x,y)\to(1,2)} 3x^2y + \frac{x}{y^3} = \cdots = 3\cdot 1^2\cdot 2 + \frac{1}{2^3} = 6 + \frac{1}{8} = \boxed{\frac{49}{8}}$

Ex $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x^3-y^3}{x-y} = \lim_{(x,y)\to(1,1)} \frac{(x-y)(x^2+xy+y^2)}{x-y} = \lim_{(x,y)\to(1,1)} (x^2+xy+y^2) = \boxed{3}$

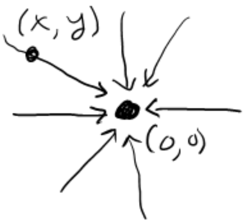
{Can't plug in (1,1), so try to cancel}

Sometimes you can exploit a familiar limit like $\displaystyle\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

Ex $\displaystyle\lim_{(x,y)\to(\pi,\pi)} \frac{\sin(x^2+y^2-2\pi^2)}{x^2+y^2-2\pi^2} = 1$ {because $x^2+y^2-2\pi^2 \to 0$}

Example $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = ?$

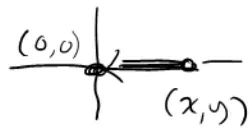
Problem: the denominator goes to 0, but nothing seems to cancel it. What to do?



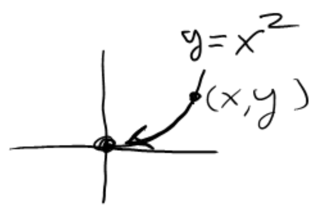
Remember: the limit should be independent of how (x,y) approaches $(0,0)$

If $(x,y)\to(0,0)$ along the x-axis (where $y=0$) we get

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2\cdot 0}{x^4+0^2} = 0$$



Same answer 0 if $(x,y)\to(0,0)$ along the y-axis.
So is the limit 0?



Now let $(x,y)\to(0,0)$ along the parabola $y=x^2$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2x^2}{x^4+(x^2)^2} = \lim_{(x,y)\to(0,0)} \frac{x^4}{2x^4} = \frac{1}{2} \neq 0$$

Conclusion LIMIT DNE

Continuity

This is a simple but significant issue. There are many useful theorems that hold only for continuous functions.

This carries over almost directly from the one-variable case

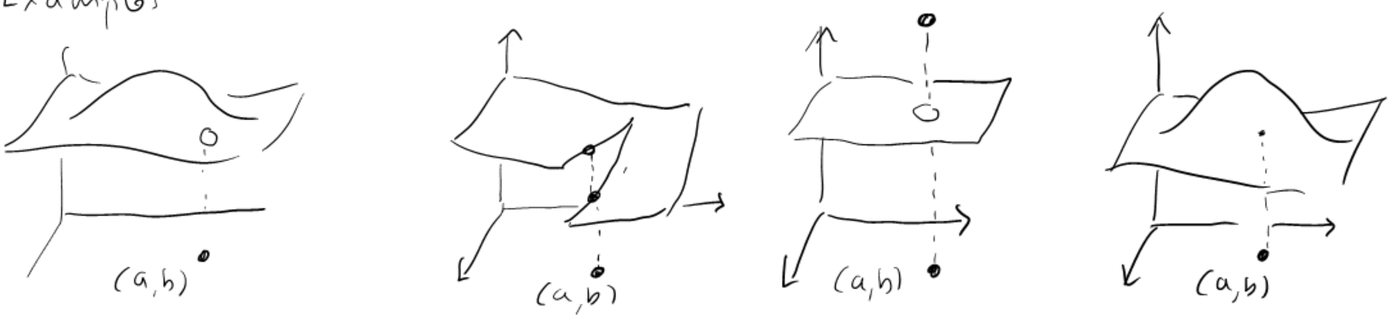
Definition A function $f(x,y)$ is continuous at (a,b) if...

1. $f(a,b)$ is defined,
2. $\lim_{(x,y)\to(a,b)} f(x,y)$ exists,
3. $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

$\Big\}$ all 3 must hold!

Function $f(x,y)$ is continuous on a region R if it's continuous at every point (a,b) in R

Examples



hole in graph ①,③ fail

tear in graph ②,③ fail

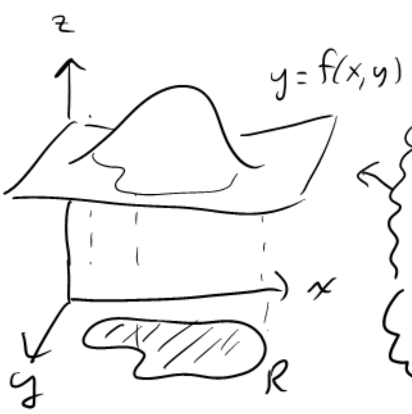
only ③ fails

①,②,③ all hold!

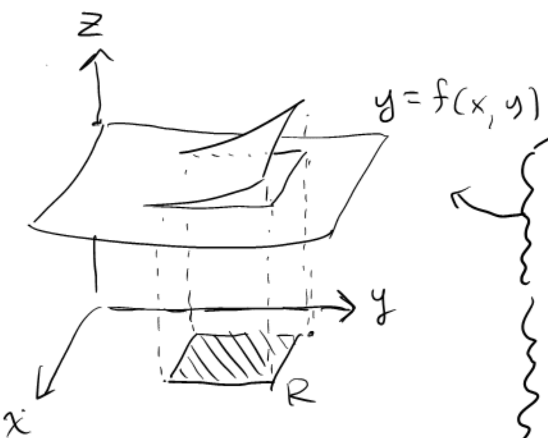
Not continuous at (a,b)

continuous at (a,b)

Low down: Continuity means no breaks holes or tears. Graph is an unbroken (if curved) sheet.



Continuous on region R



This graph is continuous on the region R. Bad stuff happens elsewhere, but not over R

These same ideas apply to functions of more than two variables. — READ THE TEXT.