Given \( y = f(x) \), we have an intuitive sense of what \( \lim_{x \to a} f(x) = L \) means.

"\( f(x) \) can be made arbitrarily close to \( L \) by choosing \( x \) sufficiently close to \( a \)."

Similarly, \( \lim_{(x,y) \to (a,b)} f(x,y) = L \)

means \( f(x,y) \) can be made arbitrarily close to \( L \) by choosing \( (x,y) \) sufficiently close to \( (a,b) \).

But this is a bit vague. For one thing, there are lots of ways for \((x,y)\) to approach \((a,b)\).

Also, what does "close" mean?

**Answer:** *Within some small distance \( \varepsilon \) or \( \delta \),

\[
\begin{align*}
(f(x,y) \text{ close to } L) & \iff (S(x,y) \text{ is within } \varepsilon \text{ units of } L) \iff |f(x,y) - L| < \varepsilon \\
(x,y) \text{ close to } (a,b) & \iff ((x,y) \text{ is within a radius of } \delta \text{ from } (a,b)) \iff \sqrt{(x-a)^2 + (y-b)^2} < \delta
\end{align*}
\]

**Precise Definition**

\( \lim_{(x,y) \to (a,b)} f(x,y) = L \) means that

for any \( \varepsilon > 0 \) (no matter how small),

there is a \( \delta > 0 \) (depending on \( \varepsilon \)) for which,

\( |f(x,y) - L| < \varepsilon \) whenever \( \sqrt{(x-a)^2 + (y-b)^2} < \delta \)

i.e., can make \( f(x,y) \)

this close to \( L \) ... ... by making \((x,y)\)

this close to \((a,b)\)
Using this definition, the usual limit rules can be proved in this more general setting:

\[
\lim_{(x,y) \to (a,b)} \frac{f(x,y)}{g(x,y)} = \lim_{(x,y) \to (a,b)} \frac{f(x,y)}{g(x,y)} \left( \text{provided both limits exist} \right)
\]

Read the complete list in the book—it should look familiar.

\[\lim_{(x,y) \to (1,1)} 3x^2y + \frac{x}{y^3} = \ldots = 3 \cdot 1^2 \cdot 1 + \frac{1}{2^3} = 6 + \frac{1}{8} = \frac{49}{8}\]

\[\lim_{(x,y) \to (1,1)} \frac{x^3 - y^3}{x-y} = \lim_{(x,y) \to (1,1)} \frac{(x-y)(x^2 + xy + y^2)}{x-y} = \lim_{(x,y) \to (1,1)} (x^2 + xy + y^2) = 3\]

\[\text{Can't plug in (1,1), so try to cancel?}\]

Sometimes you can exploit a familiar limit like \(\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1\) due to \(x^2 + y^2 - 2x^2 \to 0\)

\[\lim_{(x,y) \to (\pi, \pi)} \frac{\sin(x^2 + y^2 - 2\pi^2)}{x^2 + y^2 - 2\pi^2} = 1\quad \text{(because } x^2 + y^2 - 2\pi^2 \to 0)\]

**Example**

\[\lim_{(x,y) \to (0,0)} \frac{x^2 y}{x^4 + y^2} = ?\]

Problem: the denominator goes to 0, but nothing seems to cancel it. What to do?

Remember: the limit should be independent of how \((x,y)\) approaches \((0,0)\).

If \((x,y) \to (0,0)\) along the \(x\)-axis (where \(y=0\)) we get

\[\lim_{(x,y) \to (0,0)} \frac{x^2 y}{x^4 + y^2} = \lim_{(x,y) \to (0,0)} \frac{x^2 0}{x^4 + 0^2} = 0\]

Same answer \(0\) if \((x,y) \to (0,0)\) along the \(y\)-axis.

So is the limit \(0\)?

Now let \((x,y) \to (0,0)\) along the parabola \(y = x^2\)

\[\lim_{(x,y) \to (0,0)} \frac{x^2 y}{x^4 + y^2} = \lim_{(x,y) \to (0,0)} \frac{x^2 x^2}{x^4 + (x^2)^2} = \lim_{(x,y) \to (0,0)} \frac{x^4}{2x^4} = \frac{1}{2} \neq 0\]

**Conclusion** LIMIT DNE
Continuity

This is a simple but significant issue. There are many useful theorems that hold only for continuous functions.

This carries over almost directly from the one-variable case.

**Definition** A function \( f(x, y) \) is **continuous** at \((a, b)\) if:

1. \( f(a, b) \) is defined,
2. \( \lim_{(x, y) \to (a, b)} f(x, y) \) exists,
3. \( \lim_{(x, y) \to (a, b)} f(x, y) = f(a, b) \),

\( \text{all 3 must hold!} \)

Function \( f(x, y) \) is **continuous on a region** \( R \) if it’s continuous at every point \((a, b)\) in \( R \).

**Examples**

- Hole in graph, \( 1, 3 \) fail
- Tear in graph, \( 2, 3 \) fail
- Only \( 3 \) fails

Not continuous at \((a, b)\)

continuous at \((a, b)\)

**Low down:** Continuity means no breaks holes or tears.

Graph is an unbroken (if curved) sheet.

These same ideas apply to functions of more than two variables. — READ THE TEXT.