Section 13.2 Integrals of Vector-Valued Functions.

Recall: $\frac{d}{dt}\left[\langle f(t), g(t), h(t)\rangle\right] = \langle f'(t), g'(t), h'(t)\rangle$

Similarly $\int \langle f(t)\ g(t), h(t)\rangle dt = \left\langle \int f(t)dt, \int g(t)\,dt, \int h(t)\,dt \right\rangle$

<u>Today's Goal</u> Develop a notion of the <u>antiderivative</u> or <u>integral</u> of $\vec{r}(t)$

$$\begin{tikzcd} \vec{R}(t) \arrow[rr, bend left, "\frac{d}{dt}\ \text{differentiation}"] & & \vec{r}(t) \arrow[ll, bend left, "\text{antidifferentiation}\ \int dt"] \end{tikzcd}$$

Definition $R(t)$ is an <u>antiderivative</u> of $r(t)$ provided $\frac{d}{dt}\left[R(t)\right] = r(t)$. We express this as $R(t) = \int r(t)\,dt$.

<u>Example</u> $\int \langle t^2, 1, \cos(t)\rangle dt = \left\langle \frac{t^3}{3} + c_1, t + c_2, \sin t + c_3 \right\rangle$

$$= \left\langle \frac{t^3}{3}, t, \sin t \right\rangle + \langle c_1\ c_2\ c_3\rangle$$

$$= \left\langle \frac{t^3}{3}, t, \sin t \right\rangle + \vec{C}$$

$$\boxed{\text{In general } \int \vec{r}(t)\,dt = \vec{R}(t) + \vec{C} \text{ where } \vec{R}'(t) = \vec{r}(t)}$$

Also $\int_a^b \langle f(t), g(t), h(t)\rangle dt = \left\langle \int_a^b f(t)dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle$

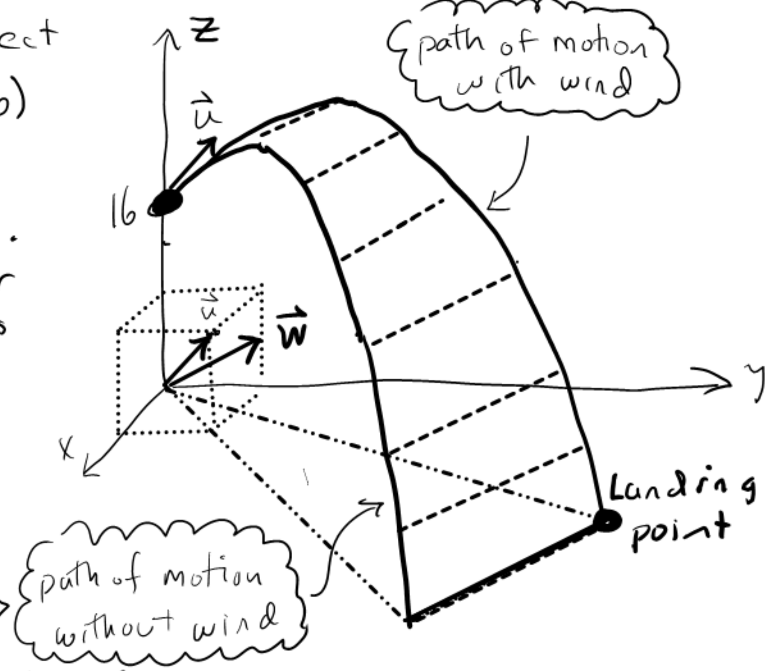
$$= \langle F(b) - F(a), G(b) - G(a), H(b) - H(a)\rangle$$

$$= \langle F(b), G(b), H(b)\rangle - \langle F(a), G(a), H(a)\rangle$$

ie $$\boxed{\int_a^b \vec{r}(t)\,dt = \vec{R}(b) - \vec{R}(a) \quad \text{where } \vec{R}(t) = \int \vec{r}(t)\,dt \quad \text{i.e. } \vec{R}'(t) = \vec{r}(t)}$$

Example At time $t=0$, object launched from point $(0,0,16)$ with a velocity of 90 ft/sec. in the direction of $\langle 1,2,2\rangle$. Wind is 10 ft/sec from S. E. [Object has no power source or engine – only acceleration is due to gravity.]

Find position and velocity functions. Where does it land?



Position $\vec{S}(t) = ?$ $\quad \vec{S}(0) = \langle 0,0,16\rangle$

Velocity $\vec{V}(t) = ?$ $\quad \vec{V}(0) = (\text{launch}) + (\text{wind})$

Accel $\vec{a}(t) = \langle 0,0,-32\rangle$ ft/sec/sec.

Wind velocity: $10\dfrac{\langle -1,1,0\rangle}{|\langle -1,1,0\rangle|} = \left\langle \dfrac{-10}{\sqrt{2}}, \dfrac{10}{\sqrt{2}}, 0\right\rangle$

Launch velocity: $90\dfrac{\langle 1,2,2\rangle}{|\langle 1,2,2\rangle|} = \langle 30,60,60\rangle$

Initial velocity: $\vec{v}(0) = (\text{launch}) + (\text{wind}) = \left\langle 30-\dfrac{10}{\sqrt{2}}, 60+\dfrac{10}{\sqrt{2}}, 60\right\rangle$

$$\vec{V}(t) = \int \vec{a}(t)\,dt = \int \langle 0,0,-32\rangle\,dt = \langle 0,0,-32t\rangle + \vec{C}$$

$$\vec{V}(0) = \langle 0,0,-32\cdot 0\rangle + \vec{C} = \left\langle 30-\frac{10}{\sqrt{2}}, 60+\frac{10}{\sqrt{2}}, 60\right\rangle$$ this is C

$$\boxed{\vec{V}(t) = \langle 0,0,-32t\rangle + \vec{C} = \left\langle 30-\frac{10}{\sqrt{2}}, 60+\frac{10}{\sqrt{2}}, 60-32t\right\rangle}$$

$$S(t) = \int \vec{V}(t)\,dt = \left\langle \left(30-\frac{10}{\sqrt{2}}\right)t, \left(60+\frac{10}{\sqrt{2}}\right)t, 60t-16t^2\right\rangle + \vec{C}$$ New C

$$\langle 0,0,16\rangle = \vec{S}(0) = \langle 0,0,0\rangle + \vec{C} \rightsquigarrow \vec{C} = \langle 0,0,16\rangle$$

$$\boxed{\vec{S}(0) = \left\langle \left(30-\frac{10}{\sqrt{2}}\right)t, \left(60+\frac{10}{\sqrt{2}}\right)t, 60t-16t^2+16\right\rangle}$$

Hits ground when $z=0$:

$$16t^2-60t-16=0$$
$$4t^2-15t-4=0$$
$$(4t+1)(t-4)=0$$

$t = -\frac{1}{4}$ sec $\qquad t = 4$ sec.

Hits ground at $t=4$ seconds

$$\boxed{\text{Lands at } \vec{S}(4) = \left\langle \left(30-\frac{10}{\sqrt{2}}\right)4, \left(60+\frac{10}{\sqrt{2}}\right)4, 0\right\rangle \approx \langle 91.63, 268.36, 0\rangle}$$