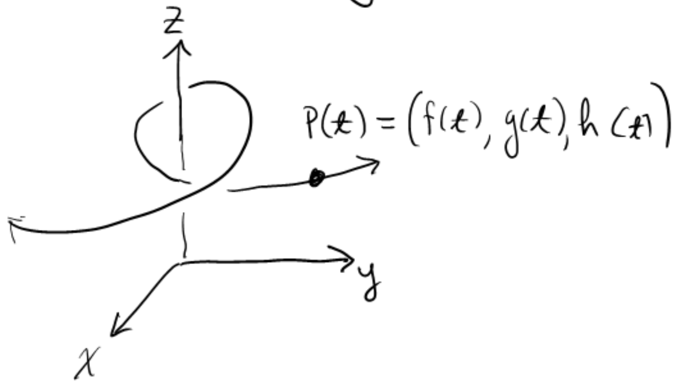
# Chapter 13 Vector-Valued Functions

## Section 13.1 Curves in Space

Suppose a point moves in space. At any time $t$ it has different $x$, $y$ and $z$ coordinates. These three coordinates vary with time $t$.

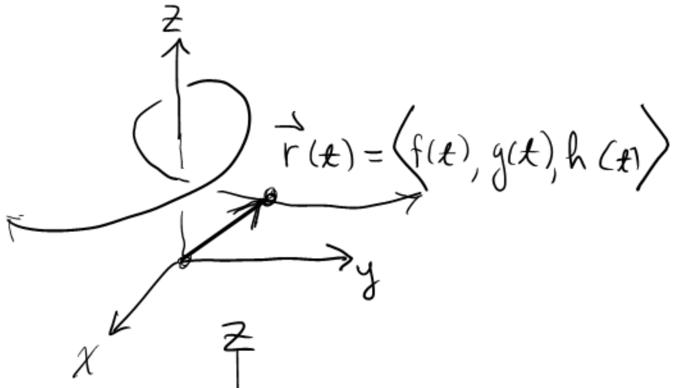
Say its position at time $t$ is

$$\left.\begin{aligned} x &= f(t) \\ y &= g(t) \\ z &= h(t) \end{aligned}\right\} a \le t \le b$$




Its trajectory is thus a curve in space that is defined parametrically.

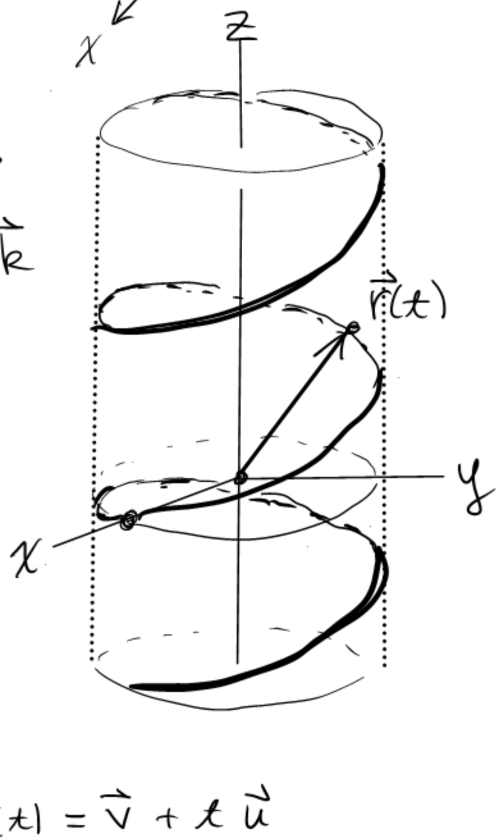
This can be expressed as a <u>vector-valued function</u> $\vec{r}(t)$ that gives a different vector for each time $t$. At time $t$ $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ points from the origen to $(f(t), g(t), h(t))$




<u>Example</u> $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$

$$= f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$$

A helix above the unit circle on The $xy$-plane.

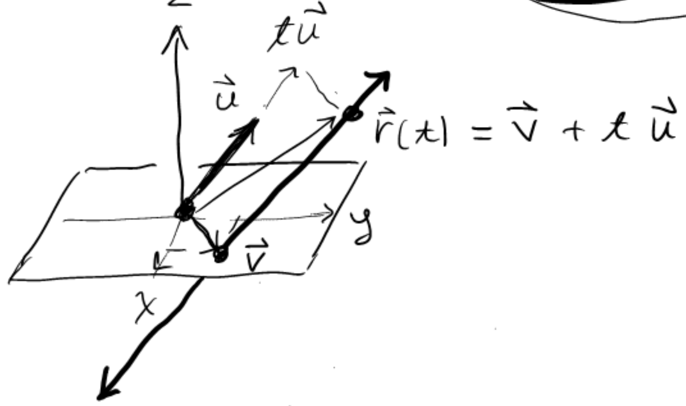



<u>Example</u>

$\vec{v} = \langle 1, 1, 0 \rangle$

$\vec{u} = \langle 0, 1, 1 \rangle$

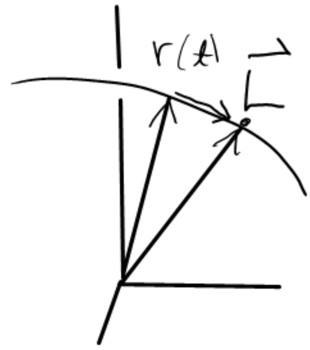
$\vec{r}(t) = \vec{v} + t\vec{u}$




This is a line through $(1,1,0)$ parallel to $\vec{u}$.

## Limits

$\lim_{t \to a} \vec{r}(t) = \vec{L}$ means $r(t)$ gets arbitrarily close to $\vec{L}$ as $t \to a$



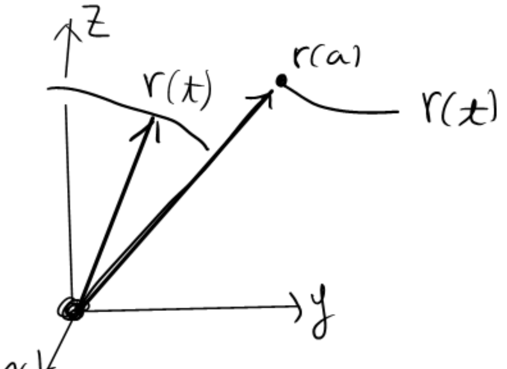

Fact: If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

then $\lim_{t \to a} \vec{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$

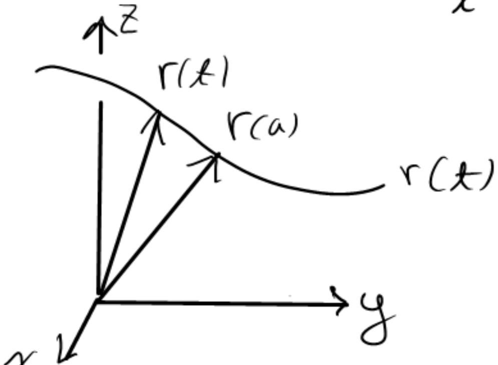
Ex $\vec{r}(t) = \langle t^2, t+3, \sqrt{t} \rangle$

$$\lim_{t \to 9} \vec{r}(t) = \left\langle \lim_{t \to 9} t^2, \lim_{t \to 9}(t+3), \lim_{t \to 9} \sqrt{t} \right\rangle = \langle 81, 12, 3 \rangle$$

Definition: $\vec{r}(t)$ is continuous at $t = a$ if $\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$



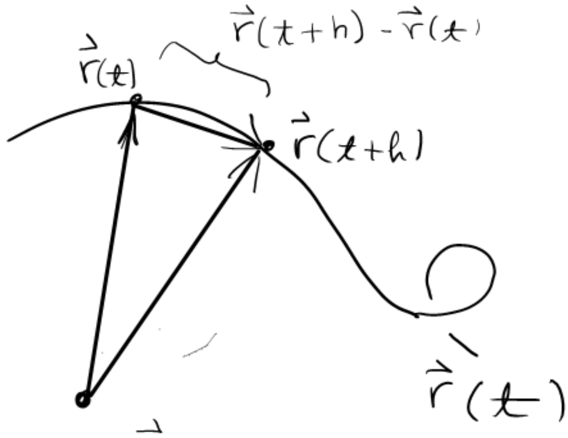

NOT Continuous at a




Continuous at a

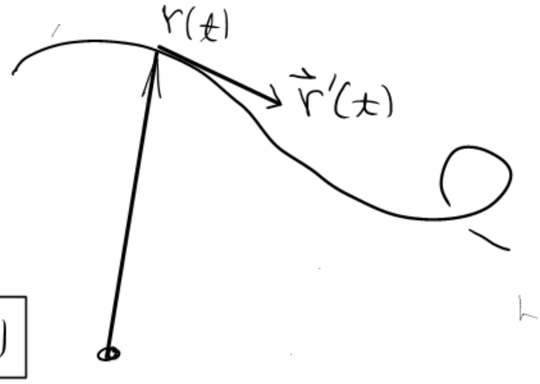
## Derivative of $\vec{r}(t)$

$$\vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$




Note: For small $h$, vector $\vec{r}(t+h) - \vec{r}(t)$ is very close to being tangent to the curve at $\vec{r}(t)$, but its very short. Dividing it by the small number $h$ scales it out to a longer vector tangent to curve.

$$\boxed{\vec{r}\,'(t) \text{ is tangent to curve at } \vec{r}(t)}$$



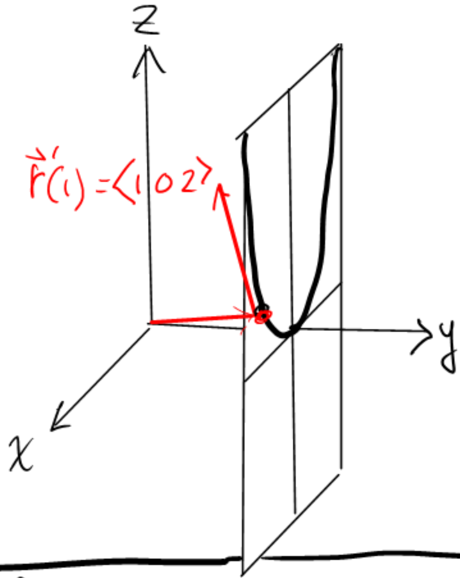
Suppose $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$$\vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \langle f'(t), g'(t), h'(t) \rangle$$

<u>Ex</u> $\vec{r}(t) = \langle t, 3, t^2 \rangle$

$\vec{r}\,'(t) = \langle 1, 0\ \ 2t \rangle$

$\vec{r}\,'(1) = \langle 1, 0, 2 \rangle$




Note that $\vec{r}\,'(1)$ is tangent to graph of $\vec{r}(t)$ at point $\vec{r}(1) = \langle 1, 3, 1 \rangle$

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<u>Notation</u> $\vec{r}\,'(t) = \frac{d}{dt}\vec{r}(t) = \frac{d}{dt}\left[\vec{r}(t)\right]$

<u>Rules</u>

- $\frac{d}{dt}\left[\vec{c}\right] = \vec{0}$
- $\frac{d}{dt}\left[c\,\vec{u}(t)\right] = c\,\vec{u}\,'(t)$
- $\frac{d}{dt}\left[f(t)\,\vec{u}(t)\right] = f'(t)\,\vec{u}(t) + f(t)\,u'(t)$
- $\frac{d}{dt}\left[\vec{u}(t) \pm \vec{v}(t)\right] = \vec{u}\,'(t) + \vec{v}\,'(t)$
- $\frac{d}{dt}\left[\vec{u}(t) \cdot \vec{v}(t)\right] = \vec{u}\,'(t) \cdot v(t) + \vec{u}(t) \cdot \vec{v}\,'(t)$
- $\frac{d}{dt}\left[\vec{u}(t) \times \vec{v}(t)\right] = \vec{u}\,'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}\,'(t)$
- $\frac{d}{dt}\left[\vec{u}(f(t))\right] = f'(t)\,\vec{u}\,'(f(t))$

This one is the derivative of a real-valued (not vector) function.

"chain rule"

Although these rules look impressive, you can often get by without them by first multiplying through and <u>then</u> differentiating.

<u>Ex</u> Find the derivative: $\vec{r}(t) = \ln(t)\,\langle t^2, 5t, t\rangle$

<u>Method I</u> $\vec{r}\,'(t) = \frac{1}{t}\langle t^2, 5t, t\rangle + \ln(t)\langle 2t, 5, 1\rangle$

$= \langle t + 2t\ln(t),\ 5 + 5\ln(t),\ 1 + \ln(t)\rangle$

<u>Method II</u> $\vec{r}(t) = \langle t^2\ln(t),\ 5t\ln(t),\ t\ln(t)\rangle$

$\vec{r}\,'(t) = \langle$ same answer as above in one step $\rangle$

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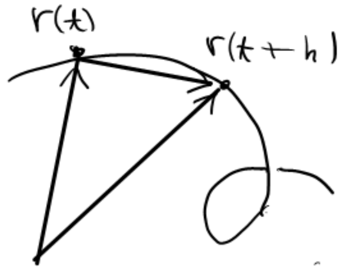
## <u>Velocity and Acceleration</u>



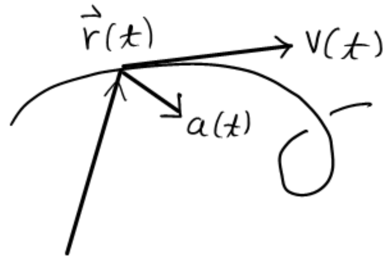

Suppose $\vec{r}(t)$ = position of object at time $t$.

$$\vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - r(t)}{h}$$

displacement

time elapsed

<u>Therefore</u>:




- velocity at time $t$ is $v(t) = \vec{r}\,'(t)$
- speed at time $t$ is $|\vec{r}\,'(t)|$
- acceleration at time $t$ is $a(t) = v'(t)$
- direction at time $t$ is $\dfrac{\vec{v}(t)}{|\vec{v}(t)|}$

<u>Example</u>

$\vec{r}(t) = \langle 0, t, t^2+2\rangle$

$\vec{v}(t) = \langle 0, 1, 2t\rangle$

$\vec{a}(t) = \langle 0, 0, 2\rangle$