Before beginning, recall the following facts. Understanding them is the key to many computations involving the orientation of lines and planes in space.

\[ \vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta \]
\[ \theta = \cos^{-1} \left( \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|} \right) \]

Vector \( \vec{u} \times \vec{v} \) is orthogonal to both \( \vec{u} \) and \( \vec{v} \). Its norm is \( |\vec{u} \times \vec{v}| = |\vec{u}| |\vec{v}| \sin \theta = \text{area of parallelogram with sides } \vec{u}, \vec{v}. \)

**Vector Equation for a Line in Space**

Equation for line \( L \) through point \( \vec{r}_0 = <x_0, y_0, z_0> \) parallel to \( \vec{v} = <v_1, v_2, v_3> \):

\[ \vec{r}(t) = \vec{r}_0 + t \vec{v} \]
\[ = <x_0 + tv_1, y_0 + tv_2, z_0 + tv_3> \]

for \( -\infty < t < \infty \).

Thus parametric form is

\[
\begin{cases}
    x = x_0 + tv_1 \\
    y = y_0 + tv_2 \\
    z = z_0 + tv_3
\end{cases}
\]

for \( -\infty \leq t \leq \infty \).

**Example** Find equation of line through points

\( \vec{r}_0 = <3, 1, 5> \) and \( \vec{r}_1 = <2, 1, 2> \)

This line passes through \( \vec{r}_0 \) and has the same direction as \( \vec{v} = \vec{r}_1 - \vec{r}_0 \) (i.e., it is parallel to \( \vec{v} \)).

Thus equation is \( \vec{r}(t) = \vec{r}_0 + t \vec{v} \)

\[ = <3, 1, 5> + t (<2, 1, 2> - <3, 1, 5>) \]
\[ = <3, 1, 5> + t (<-1, 2, -3>) = <3 - t, 1 + 2t, 5 - 3t> \]

\( \vec{r}(t) = <3 - t, 1 + 2t, 5 - 3t> \) for \( -\infty < t < \infty \).
Sometimes you will have a line through a point $P$ in the direction of a vector $\vec{v}$, and some other point $S$, and you will need to find the distance from $S$ to the line, as indicated on the right.

By trigonometry, that distance is
$$d = \left|\vec{PS}\right| \sin \theta$$
$$= \frac{\left|\vec{PS} \times \vec{v}\right|}{|\vec{v}|} = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|}$$

**Conclusion:**

**Distance from a point $S$ to a line through $P$ parallel to $\vec{v}$:**
$$\frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|}$$

Note: It's important to understand (not memorize) this formula, so if you forget it you can figure it out on the spot!

**Planes in Space:**

We can specify a plane in space with two pieces of information:

1. A point $P_0(x_0, y_0, z_0)$ on the plane
2. A vector $\vec{n} = \langle A, B, C \rangle$ normal to the plane

Given this information, what is the equation of the plane?

Any point $P(x, y, z)$ on this plane satisfies

$$\vec{n} \cdot \vec{PP_0} = 0$$

$$\langle A, B, C \rangle \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0$$

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$$

$$Ax + By + Cz = A(x_0) + B(y_0) + C(z_0)$$

$$Ax + By + Cz = D$$
Conclusion

Equation for a plane.
The plane through point \( P_0(x_0, y_0, z_0) \) and normal to \( \vec{n} = \langle A, B, C \rangle \) is the set of all points \( P(x, y, z) \) satisfying
\[
\vec{n} \cdot \vec{P_0P} = 0,
\]
or
\[
Ax + By + Cz = D
\]
where \( D = Ax_0 + By_0 + Cz_0 \).

Typically, you may have to find the equation for a plane given some data that does not include the normal vector \( \vec{n} \). In such a situation, you'll need to compute the normal from the given data, usually using the cross product.

Example. Find the equation of the plane containing the line \( \langle 2 + 3t, 2t, 3 - t \rangle \) and the point \( P_0(1, 1, 2) \).

Solution. We can get two specific points on the line as follows:
\[
\begin{align*}
t &= 0 & Q(2, 0, 3) \\
\hat{x} &= 1 & R(5, 2, 2)
\end{align*}
\]
\[
\vec{P_0Q} = \langle 1, -1, 1 \rangle \\
\vec{P_0R} = \langle 4, 1, 0 \rangle
\]

Normal: \( \vec{n} = \vec{P_0Q} \times \vec{P_0R} = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 4 & 1 & 0 \end{vmatrix} = \langle -1, 4, 5 \rangle \)

Point: \( P_0(1, 1, 2) \)

Equation: \( -x + 4y + 5z = -1 + 4(1) + 5(2) \)

\[-x + 4y + 5z = 13 \]

\[x - 4y - 5z = -13 \]

Answer. The plane consists of all points \( P(x, y, z) \) satisfying this equation.

Note: Multiplying both sides by any nonzero number (such as \(-1\)) yields an alternative (and equally valid) answer.

Read other examples in text!!
Using the dot and cross product, we can solve a variety of problems involving points, lines, and planes.

Distance $d$ from a point $S$ to a plane normal to $\vec{n}$, containing $P$.

\[
\cos \theta = \frac{\overrightarrow{PS} \cdot \vec{n}}{||\overrightarrow{PS}|| \cdot ||\vec{n}||} = \frac{d}{||\overrightarrow{PS}||} = \frac{||\overrightarrow{PS}|| \cos \theta}{||\vec{n}||} = \frac{\overrightarrow{PS} \cdot \vec{n}}{||\vec{n}||}
\]

Thus distance $= \frac{||\overrightarrow{PS} \cdot \vec{n}||}{||\vec{n}||}$

(Absolute value necessary in case dot product is negative.)

You are probably better off figuring out such problems from scratch (as above) rather than remembering the formula.

Consider two planes with normal vectors $\vec{n}_1$ and $\vec{n}_2$.

1. Vector parallel to the line of intersection is $\vec{n}_1 \times \vec{n}_2$

2. (Angle $\theta$ between the two planes) = (angle formed by normals) = $\cos^{-1}\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{||\vec{n}_1|| \cdot ||\vec{n}_2||}\right)$

Important: Hone your skills by reading examples in text and working exercises. Avoid blind use of formulas. Draw a sketch and figure the problems out using trigonometry combined with the meanings of the dot & cross products.