MATH 211 Test #1 ◇ October 5, 2017

Name: Richard R. Hammack Score: 100

Directions Except for those problems designated **short answer**, you must show and explain your work to get full credit. This test is closed-book and closed-notes. No calculators or other electronic devices. All you will need is something to write with. I will provide scratch paper.

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1. (6 points) **Short answer.**

   (a) Write the set $\{5n : n^2 \leq 16\}$ by listing its elements between braces.

   $\boxed{\{-20, -15, -10, -5, 0, 5, 10, 15, 20\}}$

   (b) Write $\{\ldots -2, 8, 18, 28, 38, 48, 58, \ldots\}$ in set-builder notation.

   $\boxed{\{8 + 10n : n \in \mathbb{Z}\}}$

   (c) Write the set $\{X \in \mathscr{P}(\mathbb{N}) : X \cap \{1,2\} = X\}$ by listing its elements between braces.

   $\boxed{\{\phi, \{1\}, \{2\}, \{1,2\}\}}$

2. (6 points) **Short answer.** Suppose A and B are sets for which $|A| = m$ and $|B| = n$. Find the following cardinalities.

   (a) $|\mathscr{P}(A)| = \boxed{2^m}$

   (b) $|\mathscr{P}(A) \times \mathscr{P}(B)| = 2^m 2^n = \boxed{2^{m+n}}$

   (c) $|\{X \in \mathscr{P}(B) : |X| = 5\}| = \boxed{\binom{n}{5}}$ (This is the number of subsets X of B that have cardinality 5)

3. (8 points) **Short answer.** Suppose $A = \{1, 3, 4, 6, 9\}$ and $B = \{4, 5, 6, 8, 9\}$ are two sets in a universal set $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

   (a) $B - A = \boxed{\{5, 8\}}$

   (b) $A \cap \overline{B} = \{1, 3, 4, 6, 9\} \cap \{1, 2, 3, 7\} = \boxed{\{1, 3\}}$

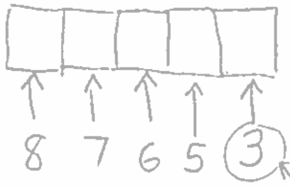
   (c) $\overline{U} = \boxed{\phi}$

   (d) $(B - A)^2 = (B-A) \times (B-A) = \{5, 8\} \times \{5, 8\}$

   $= \boxed{\{(5,5), (5,8), (8,5), (8,8)\}}$

4. (20 points) This question concerns **length-5** lists made from the letters A, B, C, D, E, F, G, H, I.

(a) How many such lists have no repetition and end with a vowel?

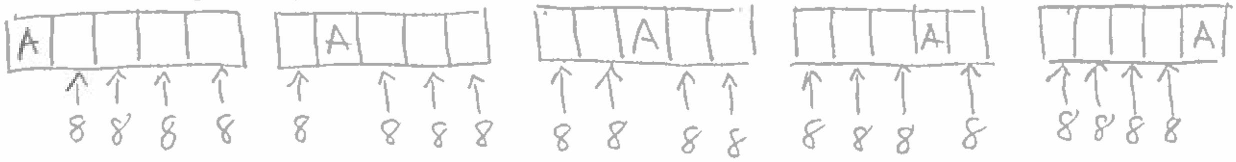


(fill this first: 3 vowels, A, E, I)

Ans. $8\cdot 7\cdot 6\cdot 5\cdot 3 = \boxed{5040}$

(b) How many such lists are there if repetition is allowed, and the list contains at *exactly* one A?

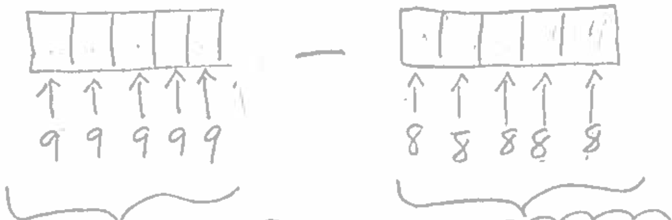
There are 5 types of lists.



By addition principle, answer is

$$5\cdot 8^4 = \boxed{20,480}$$

(c) How many such lists are there if repetition is allowed, and the list contains at least one A?



repetition OK any letter

repetition OK, No A's

Answer:
By subtraction principle

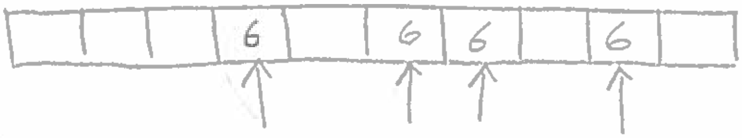
$$9^5 - 8^5 = \boxed{2681}$$

(d) How many such lists are there that have no repetition and are in alphabetical order?

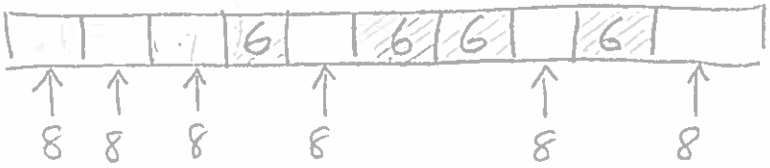
Answer: $\binom{9}{5} = \boxed{126}$

(To make such a list, just pick 5 of the 9 letters. Then put them in alphabetical order)

5. (10 points) How many 10-digit integers contain no 0's and exactly four 6's?



First choose 4 out of 10 positions for the 6's. There are $\binom{10}{4}$ ways to do this.



Then fill in remaining 6 positions with a choice of 8 digits for each (anything but 0 & 6)

Answer: $\boxed{\binom{10}{4} 8^6 \text{ lists}}$

6. (10 points) How many non-negative integer solutions does the equation $x + y + z = 50$ have?

$$\overbrace{* * * \cdots *}^{x \text{ stars}} \mid \overbrace{* * * \cdots *}^{y \text{ stars}} \mid \overbrace{* * * * \cdots *}^{z \text{ stars}}$$

list, length 52 has 50 stars and 2 bars.

Number of such lists is $\binom{52}{2}$.

Answer $\binom{52}{2} = \frac{52 \cdot 51}{2} = 1326$ solutions.

7. (10 points)

(a) Here are the first several rows of Pascal's triangle. Write the next row.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

(b) Use part (a) to find the coefficient of x^2y^4 in $(2x-y)^6$.
Please give the exact (i.e., worked out) value.

Relevant term is $\binom{6}{4}(2x)^2(-y)^4$

$= 15\, 2^2 x^2 y^4 = 60x^2y^4$

Thus coefficient is $\boxed{60}$

8. (10 points) Use the binomial theorem to show that
$10^n = 9^0\binom{n}{0} + 9^1\binom{n}{1} + 9^2\binom{n}{2} + 9^3\binom{n}{3} + 9^4\binom{n}{4} + \cdots + 9^n\binom{n}{n}$.

By binomial Theorem,

$$10^n = (1+9)^n$$

$$= \binom{n}{0}1^n9^0 + \binom{n}{1}1^{n-1}9^1 + \binom{n}{2}1^{n-2}9^2 + \cdots + \binom{n}{n}1^0 9^n$$

$$= 9^0\binom{n}{0} + 9^1\binom{n}{1} + 9^2\binom{n}{2} + 9^3\binom{n}{3} + \cdots + 9^n\binom{n}{n}$$

9. (10 points) A department consists of 7 men and 9 women. From this department you select a committee with 3 men and 2 women. In how many ways can you do this?

$$\binom{7}{3}\cdot\binom{9}{2} = \frac{7!}{3!\,4!}\;\frac{9!}{2!\,7!}$$

↑ Choose 3 men

↑ choose 2 women

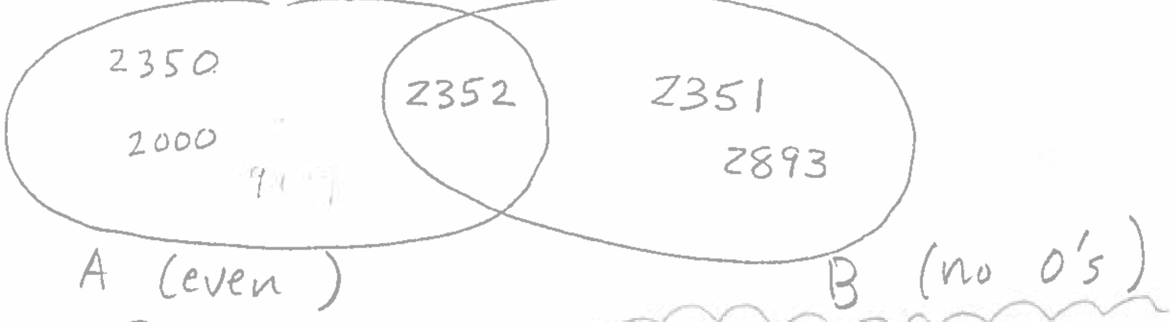
$$= \frac{9!}{3!\;4!\;2!}$$

$$= \frac{9\cdot \overset{4}{\cancel{8}}\cdot 7\cdot 6\cdot 5\cdot \cancel{4!}}{3\cdot 2\cdot 1\cdot \cancel{4!}\cdot 2}$$

$$= 9\cdot 4\cdot 7\cdot 5 = \boxed{1260 \text{ ways}}$$

10. (10 points) How many 4-digit positive integers are there that are even or contain no 0's?

Divide these integers into two sets, as follows:



Numbers in A: ☐☐☐☐

↑ ↑ ↑ ↑

9 10 10 5

Thus $|A| = 9\cdot 10\cdot 10\cdot 5$
$= 4500$

Numbers in B: ☐☐☐☐

↑ ↑ ↑ ↑

9 9 9 9

Thus $|B| = 9^4 = 6561$

Also a number in $A\cap B$ looks like ☐☐☐☐

↑ ↑ ↑ ↑

9 9 9 4

Thus $|A\cap B| = 9^3\cdot 4 = 2916$.

By inclusion-exclusion principle, the final answer is $|A\cup B| = |A|+|B|-|A\cap B| = 4500 + 9^4 - 9^3\cdot 4$

$= \boxed{8145}$