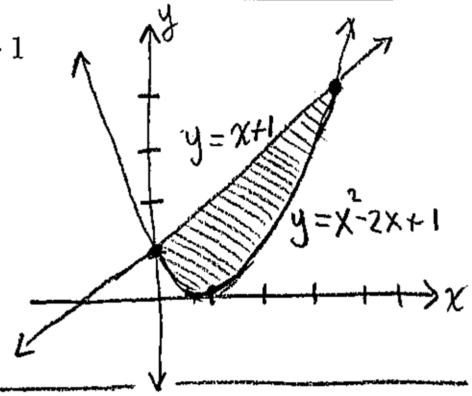
Name: Richard **TEST 1 ♡** MATH 201

February 13, 2024

1. Find the area of the region bounded by $y = x^2 - 2x + 1$ and $y = x + 1$



First find the intersection points.

$$x^2 - 2x + 1 = x + 1$$
$$x^2 - 3x = 0$$
$$x(x-3) = 0$$
$$x = 0 \qquad x = 3$$

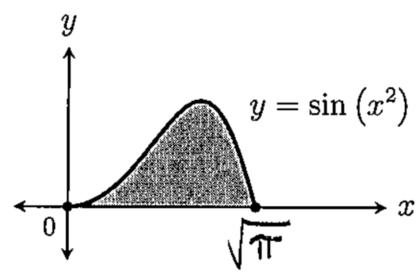
The graphs intersect at (0,1) and (3,4)

$$A = \int_0^3 \left((x+1) - (x^2 - 2x + 1)\right) dx = \int_0^3 (-x^2 + 3x)\, dx$$

$$= \left[-\frac{x^3}{3} + 3\frac{x^2}{2}\right]_0^3 = \left(\frac{-3^3}{3} + \frac{3 \cdot 3^2}{2}\right) - \left(-\frac{0^3}{3} + 3\frac{0^2}{2}\right)$$

$$= \frac{-27}{3} + \frac{27}{2} = -9 + \frac{27}{2} = \frac{-18}{2} + \frac{27}{2} = \boxed{\frac{9}{2} \text{ square units}}$$

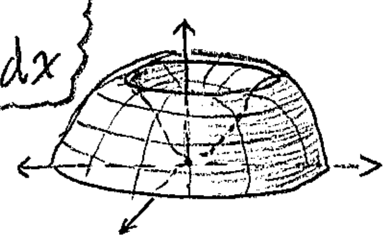
2. The shaded region below is rotated around the y-axis. Find the volume of the resulting solid.



By shells:

$$V = \int_0^{\sqrt{\pi}} 2\pi x\, f(x)\, dx$$

$$= \pi \int_0^{\sqrt{\pi}} \sin(x^2)\, 2x\, dx \qquad \left\{ \begin{array}{l} u = x^2 \\ du = 2x\,dx \end{array} \right\}$$

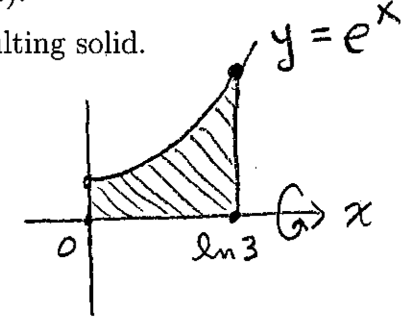


$$= \pi \int_{0^2}^{\sqrt{\pi}^2} \sin(u)\, du$$

$$= \pi\left[-\cos(u)\right]_0^{\pi} = \pi\left(-\cos(\pi) - (-\cos(0))\right) = \pi(1+1)$$

$$= \boxed{2\pi \text{ cubic units}}$$

3. Consider the region bounded by $y = e^x$, $y = 0$, $x = 0$ and $x = \ln(3)$. This region is rotated around the x-axis. Find the volume of the resulting solid.



Volume by slicing:

As shown on the right,

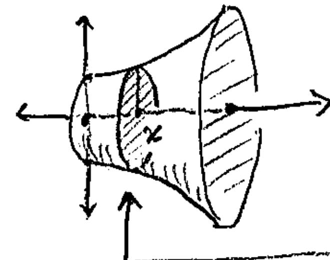
$$A(x) = \pi e^{2x}$$

$$V = \int_0^{\ln(3)} \pi e^{2x}\,dx = \pi \left[\frac{e^{2x}}{2}\right]_0^{\ln(3)}$$

$$= \frac{\pi}{2}\left(e^{2\ln(3)} - e^{2\cdot 0}\right)$$

$$= \frac{\pi}{2}\left(e^{\ln(3^2)} - 1\right) = \frac{\pi}{2}(3^2 - 1)$$

$$= \boxed{4\pi \text{ cubic units}}$$



$$\boxed{\begin{aligned} A(x) &= \pi (e^x)^2 \\ &= \pi e^{2x} \end{aligned}}$$

4. The graph of $y = x^3$ for $0 \le x \le 1$ is rotated around the $\cancel{x}$-axis. Find the area of the resulting surface.

$$\int_0^1 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx = \int_0^1 2\pi x^3 \sqrt{1 + (3x^2)^2}\,dx$$

$$= 2\pi \int_0^1 \sqrt{1 + 9x^4}\; x^3\,dx$$

$$\begin{aligned} u &= 1 + 9x^4 \\ du &= 36x^3\,dx \\ x^3\,dx &= \frac{1}{36}\,du \end{aligned}$$

$$= 2\pi \int_{1+9\cdot 0^4}^{1+9\cdot 1^4} \sqrt{u}\;\frac{1}{36}\,du$$

$$= \frac{\pi}{18}\int_1^{10} \sqrt{u}\,du = \frac{\pi}{18}\left[\frac{u^{1/2+1}}{1/2+1}\right]_1^{10} = \frac{\pi}{18}\left[\frac{2\sqrt{u}^{\,3}}{3}\right]_1^{10}$$

$$= \frac{\pi}{27}\left(\sqrt{10}^{\,3} - \sqrt{1}^{\,3}\right) = \boxed{\frac{\pi}{27}\left(10\sqrt{10} - 1\right) \text{ square units}}$$

5. Find the arc length of the curve $y = \dfrac{2\sqrt{x}^3}{3}$ from $x = 0$ to $x = 8$.

$$y = \frac{2}{3}x^{3/2}$$
$$y' = x^{1/2} = \sqrt{x}$$

$$L = \int_0^8 \sqrt{1 + (f'(x))^2}\, dx$$

$$= \int_0^8 \sqrt{1 + \sqrt{x}^2}\, dx = \int_0^8 \sqrt{1+x}\, dx$$

$$u = 1+x$$
$$du = dx$$

$$= \int_{1+0}^{1+8} \sqrt{u}\, du = \int_1^9 u^{1/2}\, du = \left[\frac{u^{1/2+1}}{1/2+1}\right]_1^9$$

$$= \left[\frac{2\sqrt{u}^3}{3}\right]_1^9 = \frac{2}{3}\left(\sqrt{9}^3 - \sqrt{1}^3\right) = \frac{2}{3}(27-1)$$

$$= \boxed{\frac{52}{3} \text{ units}}$$

6. A variable force moves an object from $\ln(\pi/4)$ to $\ln(\pi/2)$ on the number line (units in meters). At any point x between $\ln(\pi/4)$ and $\ln(\pi/2)$, the force is $e^x \cos(e^x)$ Newtons. Find the work done in moving the object from $\ln(\pi/4)$ to $\ln(\pi/2)$.

$$W = \int_{\ln(\pi/4)}^{\ln(\pi/2)} e^x \cos(e^x)\, dx$$

$$u = e^x$$
$$du = e^x dx$$

$$= \int_{e^{\ln(\pi/4)}}^{e^{\ln(\pi/2)}} \cos(u)\, du = \int_{\pi/4}^{\pi/2} \cos(u)\, du$$

$$= \left[\sin(u)\right]_{\pi/4}^{\pi/2} = \sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{4}\right)$$

$$= \boxed{1 - \frac{\sqrt{2}}{2} \text{ J}}$$