

Section 8.4 Trigonometric Substitutions

Motivational problem: $\int \sqrt{1-x^2} dx = ?$

Familiar substitutions and integration by parts don't work.

Idea: Try trig substitution that eliminates the radical.

$$\int \sqrt{1-x^2} dx = \int \sqrt{1-\sin^2\theta} \cos\theta d\theta = \int \cos\theta \cos\theta d\theta$$

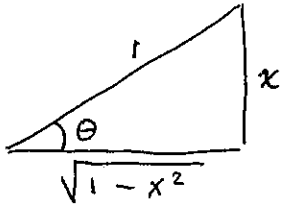
$$x = \sin\theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

$$dx = \cos\theta d\theta$$

$$\theta = \sin^{-1}(x)$$

$$= \int \cos^2\theta d\theta = \frac{1}{2}\theta + \frac{1}{2}\sin\theta \cos\theta + C$$

$$= \frac{1}{2}\sin^{-1}(x) + \frac{1}{2}x\sqrt{1-x^2} + C$$



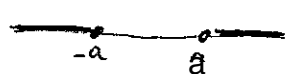
Check: $\frac{d}{dx} \left[\frac{1}{2}\sin^{-1}(x) + \frac{1}{2}x\sqrt{1-x^2} + C \right]$

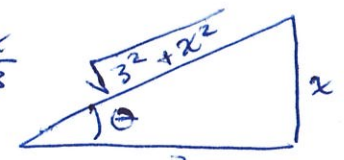
$$= \frac{1}{2} \frac{1}{\sqrt{1-x^2}} + \frac{1}{2}\sqrt{1-x^2} + \frac{1}{2}x \frac{-2x}{2\sqrt{1-x^2}} + 0$$

$$= \frac{1}{2\sqrt{1-x^2}} + \frac{\sqrt{1-x^2}\sqrt{1-x^2}}{2\sqrt{1-x^2}} - \frac{2x^2}{2\sqrt{1-x^2}} = \frac{2-2x^2}{2\sqrt{1-x^2}} = \frac{1-x^2}{\sqrt{1-x^2}} = \sqrt{1-x^2}$$

Simplifications:

Expression	Substitution	Result
$\sqrt{a^2 - x^2}$	$x = a \sin\theta \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$\sqrt{a^2 - a^2 \sin^2\theta} = a \cos\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$\sqrt{a^2 + a^2 \tan^2\theta} = a \sec\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta \quad \begin{cases} 0 \leq \theta < \frac{\pi}{2} \text{ if } x \geq a \\ \frac{\pi}{2} < \theta \leq \pi \text{ if } x \leq -a \end{cases}$	$\sqrt{a^2 \sec^2\theta - a^2} = \sqrt{\tan^2\theta} = \tan\theta $



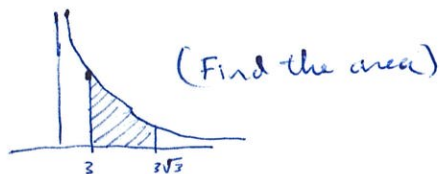
$$\text{Ex } \int \frac{1}{x\sqrt{9+x^2}} dx \quad x = 3 \tan \theta \quad dx = 3 \sec^2 \theta d\theta \quad \tan \theta = \frac{x}{3}$$


$$= \int \frac{3 \sec^2 \theta d\theta}{3 \tan \theta \sqrt{9 + (3 \tan \theta)^2}} = \int \frac{\sec^2 \theta d\theta}{\tan \theta \cdot 3 \sec \theta} d\theta = \frac{1}{3} \int \frac{\sec \theta}{\tan \theta} d\theta$$

$$= \frac{1}{3} \int \csc \theta d\theta = -\frac{1}{3} \ln |\csc \theta + \cot \theta| + C = \boxed{-\frac{1}{3} \ln \left| \frac{\sqrt{9+x^2}}{x} + \frac{3}{x} \right|}$$

$$\text{Ex } \int_3^{3\sqrt{3}} \frac{1}{x\sqrt{9+x^2}} dx = -\frac{1}{3} \ln \left| \frac{\sqrt{9+x^2}}{x} + \frac{3}{x} \right|_3^{3\sqrt{3}} =$$

$$-\frac{1}{3} \ln \left| \frac{\sqrt{4+27}}{3\sqrt{3}} + \frac{3}{3\sqrt{3}} \right| + \frac{1}{3} \ln \left| \frac{\sqrt{9+9}}{3} + \frac{3}{3} \right| = \boxed{-\frac{1}{3} \ln \left| \frac{e}{\sqrt{3}} + \frac{1}{\sqrt{3}} \right| + \frac{1}{3} \ln |\sqrt{2} + 1|}$$



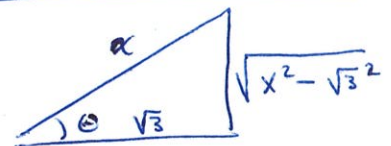
$$\text{Ex } \int_3^{3\sqrt{3}} \frac{1}{x\sqrt{9+x^2}} dx = \frac{1}{3} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \csc \theta d\theta = -\frac{1}{3} \ln |\csc \theta + \cot \theta| \Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= -\frac{1}{3} \ln \left| \csc \frac{\pi}{3} + \cot \frac{\pi}{3} \right| + \frac{1}{3} \ln \left| \csc \frac{\pi}{4} + \cot \frac{\pi}{4} \right|$$

$$= \boxed{\frac{1}{3} \ln \left| \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} \right| + \frac{1}{3} \ln |\sqrt{2} + 1|}$$

$$\text{Ex } \int \frac{dx}{x^4 \sqrt{x^2-3}} \quad x = \sqrt{3} \sec \theta \quad dx = \sqrt{3} \sec \theta \tan \theta d\theta \quad \sec \theta = \frac{x}{\sqrt{3}}$$

$$\sec \theta = \frac{x}{\sqrt{3}}$$



$$= \int \frac{\sqrt{3} \sec \theta \tan \theta d\theta}{(\sqrt{3} \sec \theta)^4 ((\sqrt{3} \sec \theta)^2 - 3)} = \int \frac{\sqrt{3} \sec \theta \tan \theta d\theta}{9 \sec^4 \theta \sqrt{3} \tan \theta} = \frac{1}{9} \int \frac{d\theta}{\sec^3 \theta} = \frac{1}{9} \int \cos^3 \theta d\theta$$

$$\frac{1}{9} \int \cos^2 \theta \cos \theta d\theta = \frac{1}{9} \int (1 - \sin^2 \theta) \cos \theta d\theta = \frac{1}{9} \int (1 - u^2) du$$

$$= \frac{1}{9} \left(u - \frac{u^3}{3} \right) + C = \frac{1}{9} \sin \theta - \frac{1}{27} \sin^3 \theta + C = \boxed{\frac{1}{9} \frac{\sqrt{x^2-3}}{x} - \frac{1}{27} \left(\frac{\sqrt{x^2-3}}{x} \right)^3 + C}$$

Notice that some familiar formulas fall out of this.

Formulas:

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1}\left|\frac{u}{a}\right| + C$$

Reason: $u = a \sin \theta$
 $du = a \cos \theta d\theta$

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \int \frac{a \cos \theta d\theta}{\sqrt{a^2 - (a \sin \theta)^2}} = \int \frac{a \cos \theta d\theta}{a \cos \theta}$$

$$= \int d\theta = \theta + C = \sin^{-1}\left(\frac{u}{a}\right) + C$$

So if you understand how to use this technique, you don't need to remember these formulas.

Integrals involving $ax^2 + bx + c$

$$\int \frac{dx}{\sqrt{5 + 6x - x^2}} = \int \frac{dx}{\sqrt{14^2 - (x-3)^2}} = \int \frac{dx}{\sqrt{14^2 - u^2}} = \sin^{-1}\left(\frac{u}{14}\right) + C$$

$$5 + 6x - x^2$$

$$= -(x^2 - 6x) + 5$$

$$= -(x^2 - 6x + 9) + 14$$

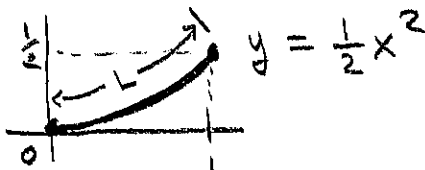
$$= -(x-3)^2 + 14$$

$$= \sqrt{14^2 - (x-3)^2}$$

$$u = x - 3$$

$$du = dx$$

$$= \boxed{\sin^{-1}\left(\frac{x-3}{14}\right) + C}$$

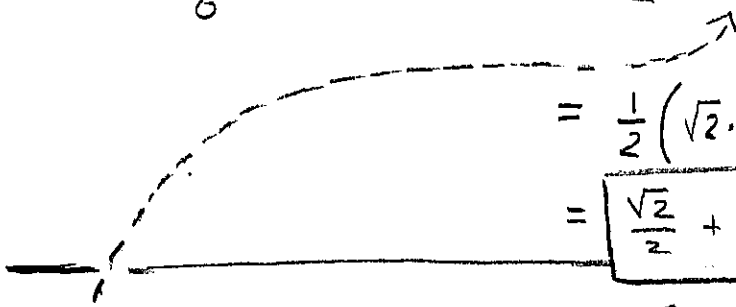
Ex Find the length:  $y = \frac{1}{2}x^2$

$$L = \int_0^1 \sqrt{1 + (f'(x))^2} dx = \int_0^1 \sqrt{1 + x^2} dx$$

$x = \tan \theta$
 $dx = \sec^2 \theta d\theta$

$$= \int_{\tan^{-1}(0)}^{\tan^{-1}(1)} \sqrt{1 + \tan^2 \theta} \sec^2 \theta d\theta = \int_0^{\pi/4} \sqrt{\sec^2 \theta} \sec^2 \theta d\theta$$

$$= \int_0^{\pi/4} \sec^3 \theta d\theta = \left[\frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) \right]_0^{\pi/4}$$



$$= \frac{1}{2} (\sqrt{2} \cdot 1 + \ln |\sqrt{2} + 1|) - \frac{1}{2} (1 \cdot 0 + \ln |1 + 0|)$$

$= \frac{\sqrt{2}}{2} + \ln |\sqrt{2} + 1| \text{ units}$

$$\int \sec^3 \theta d\theta = \sec \theta \tan \theta - \int \sec \theta \tan^2 \theta d\theta$$

$u = \sec \theta \quad dv = \sec^2 \theta d\theta$
 $du = \sec \theta \tan \theta d\theta \quad v = \tan \theta$

$$= \sec \theta \tan \theta - \int \sec \theta (\sec^2 \theta - 1) d\theta$$

$$= \sec \theta \tan \theta - \int \sec^3 \theta - \sec \theta d\theta$$

$$= \sec \theta \tan \theta - \int \sec^3 \theta d\theta + \int \sec \theta d\theta$$

$$2 \int \sec^3 \theta d\theta = \sec \theta \tan \theta + \int \sec \theta d\theta$$

$$\int \sec^3 \theta d\theta = \frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|)$$