§11.2 Power Series

Definitions

- Power series centered at a:

$$\sum_{k=0}^{\infty} c_k (x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

- Power series centered at 0

$$\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

Examples

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(x-1)^k = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots$$

← Taylor series for $\ln(x)$ centered at 1

$$\sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots$$

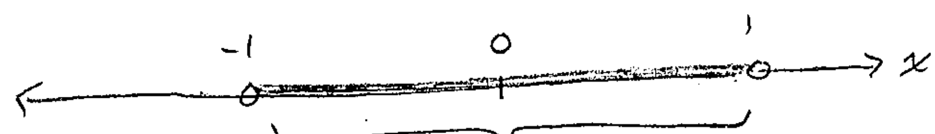
← Maclaurin series for e^x

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$$

← Geometric series $a=1$ $r=x$ Maclaurin series for $\frac{1}{1-x}$

When you plug in a value for x in a power series you get a particular infinite series. The series may converge for some x and diverge for others.

Ex $\sum_{k=0}^{\infty} x^k$
$$\begin{cases} x=2: & 1+2+2^2+2^3+2^4+\cdots \quad \text{diverges} \\ x=\frac{1}{2}: & 1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\cdots \quad \text{converges} \end{cases}$$



Interval of convergence for $\sum x^k$. Series converges for values of x in this interval.

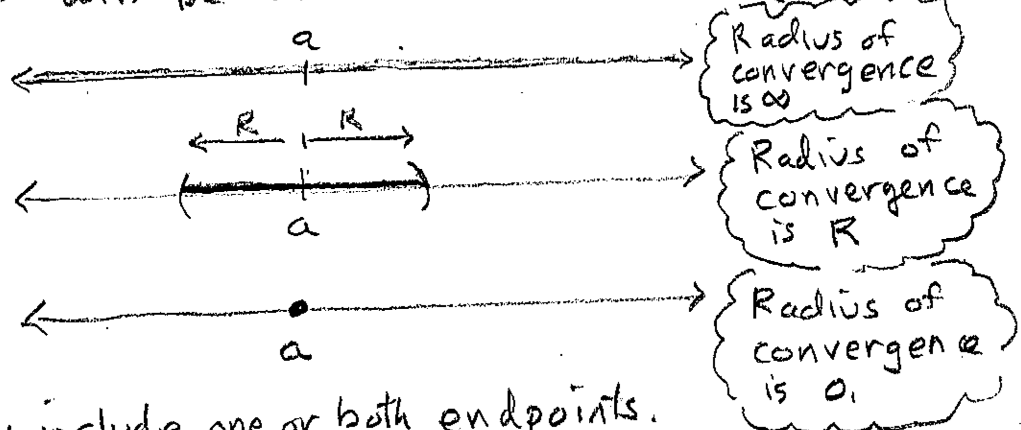
<u>Note</u> $\sum c_k(x-a)^k$ <u>always</u> converges for $x=a$.
Possibly converges for other values of x too!

<u>Fact</u> For any power series $\sum c_k(x-a)^k$, the interval of convergence will be one of the following

Ⓐ $(-\infty, \infty)$

Ⓑ $(a-R, a+R)$

Ⓒ Only $x=a$



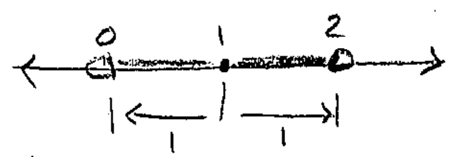
<u>Note</u> Case Ⓑ may include one or both endpoints.

<u>Example</u> Taylor series for $\ln(x)$ centered at 1 is $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(x-1)^k$.
Find its interval of convergence.

<u>Ratio Test:</u> $\displaystyle\lim_{k\to\infty}\left|\frac{\frac{(-1)^k}{k+1}(x-1)^{k+1}}{\frac{(-1)^{k-1}}{k}(x-1)^k}\right| = \lim_{k\to\infty}\left|\frac{k}{k+1}(x-1)\right| = |x-1|.$

Series converges for $|x-1| < 1 \Rightarrow -1 < x-1 < 1$
$\Rightarrow 0 < x < 2$

Radius of convergence: $R=1$.



<u>What about $x=0$?</u> $\sum \frac{(-1)^{k-1}}{k}(0-1)^k = \sum \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$

Harmonic series – <u>diverges</u>.

<u>What about $x=2$?</u> $\sum \frac{(-1)^{k-1}}{k}(2-1)^k = \sum \frac{(-1)^{k-1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} +$

Alternating harmonic series – converges!

<u>True interval of convergence</u> $(0, 2]$

<u>Unanswered question</u> Does series converge to $\ln(x-1)$ on this interval? Stay tuned.

Ex Maclaurin Series for $\cos(x)$: $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k}$

Find interval of convergence.

Ratio Test $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{\frac{x^{2(k+1)}}{(2(k+1))!}}{\frac{x^{2k}}{(2k)!}} = \lim_{k\to\infty}\frac{(2k)!}{(2k+2)!}\frac{x^{2k+2}}{x^{2k}}$

$= \lim_{k\to\infty}\frac{(2k)!\,x^{2k+2-2k}}{(2k+2)(2k+1)(2k)!} = \lim_{k\to\infty}\frac{x^2}{(2k+2)(2k+1)} = 0 < 1$

This series converges for all x. Interval of convergence: $(-\infty,\infty)$. Radius of convergence is $R=\infty$.

Ex: Find the interval of convergence for $\sum_{k=1}^{\infty} k!x^k$

Ratio Test $\lim_{k\to\infty}\left|\frac{(k+1)!\,x^{k+1}}{k!\;x^k}\right| = \lim_{k\to\infty}(k+1)x = \begin{cases}0 & \text{if } x=0\\ \infty & \text{if } x\neq 0\end{cases}$

Interval of convergence:  Radius of convergence: $R=0$

Theorem 11.4

① $\sum c_kx^k + \sum d_kx^k = \sum (c_k+d_k)x^k$

② $x^m\sum c_kx^k = \sum c_kx^{m+k}$

} Assuming convergence.

③ If $f(x)=\sum c_kx^k$, then $f(bx^m)=\sum c_k(bx^m)^k=\sum c_kb^kx^{mk}$

Ex $\frac{x^3}{1-x} = x^3\frac{1}{1-x} = x^3\sum_{k=0}^{\infty}x^k = \sum_{k=0}^{\infty}x^{k+3} = x^3+x^4+x^5+\ldots$

Ex $\cos(x) = \sum\frac{x^{2k}}{(2k)!}$

$\cos(\pi x^2) = \sum\frac{(\pi x^2)^{2k}}{2k!} = 1-\frac{(\pi x^2)^2}{2!}+\frac{(\pi x^2)^3}{3!}-\ldots = 1-\frac{\pi^2x^4}{2!}+\frac{\pi^3x^6}{3!}+\ldots$

Theorem 11.5

Suppose $f(x) = \sum c_k (x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$ on some interval I.

① $f'(x) = \sum k c_k (x-a)^{k-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$

② $\int f(x)\,dx = \sum \frac{c_k (x-a)^{k+1}}{k+1} + C = \left(c_0 x + \frac{c_1(x-a)^2}{2} + \frac{c_2(x-a)^3}{3} + \cdots\right) + C$

Ex $f(x) = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ on $(-1,1)$.

$f'(x) = \frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} k x^{k-1}$

$\therefore \quad \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^5 + \cdots$ on $(-1,1)$

Ex $f(x) = \underbrace{1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots}_{\text{geometric series, } a=1,\ r=-x^2} = \frac{1}{1+x^2}$ on $(-1,1)$.

$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots$

$\int \frac{1}{1+x^2}\,dx = \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots\right) + C$

$\tan^{-1}(x) = \left(x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right) + C$

Plug in $x = 0$. Get $C = 0$.

$\boxed{\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} \cdots}$ on $[-1,1]$

Consequence $\frac{\pi}{4} = \tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11}$

$\therefore \boxed{\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \cdots}$