§10.7 The Ratio and Root Tests

Here are two more tests to determine if a series converges.

Theorem 10.20 (Ratio Test)

Given a series $\sum a_k$, suppose $\lim\limits_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = r$.

1. If $r<1$, the series converges absolutely
2. If $r>1$, the series diverges
3. If $r=1$, the test is inconclusive

Why it works. Suppose $\lim\limits_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = r$. Then

$\left|\frac{a_{k+1}}{a_k}\right| = \frac{|a_{k+1}|}{|a_k|} \approx r$ for large k.

So $|a_{k+1}| \approx r|a_k|$.

Hence $\sum |a_k|$ acts like a geometric series with ratio r.

The case $r=1$ is inconclusive because, for instance

- $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ <u>diverges</u> but $\lim\limits_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim\limits_{k\to\infty}\frac{\frac{1}{k+1}}{\frac{1}{k}} = 1$.

- $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ <u>converges</u> but $\lim\limits_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = 1$.

Example Does $\sum\limits_{k=1}^{\infty}\frac{5^k}{k!}$ converge?

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{5^{k+1}/(k+1)!}{5^k/k!} = \lim_{k\to\infty}\frac{5^{k+1}}{(k+1)!}\,\frac{k!}{5^k}$$

$$= \lim_{k\to\infty}\frac{5}{k+1} = \boxed{0}$$

Series converges by ratio test!

Ex Does $\sum_{k=1}^{\infty} \frac{2(-1)^k}{k^3+e^k}$ converge?

$$\lim_{k\to\infty} \left| \frac{\frac{2(-1)^{k+1}}{(k+1)^3+e^{k+1}}}{\frac{2(-1)^k}{k^3+e^k}} \right| = \lim_{k\to\infty} \frac{\frac{2}{(k+1)^3+e^{k+1}}}{\frac{2}{k^3+e^k}}$$

$$= \lim_{k\to\infty} \frac{k^3+e^k}{(k+1)^3+e^{k+1}} = \lim_{k\to\infty} \frac{3k^2+e^k}{3(k+1)^2+e^{k+1}}$$

$$= \lim_{k\to\infty} \frac{6k+e^k}{6(k+1)+e^{k+1}} = \lim_{k\to\infty} \frac{6+e^k}{6+e^{k+1}}$$

L'Hopital $\times 4$

$$= \lim_{k\to\infty} \frac{e^k}{e^{k+1}} = \frac{1}{e} < 1$$

Ans: $\boxed{\text{It converges!}}$

Theorem 10.21 (Root Test)

Given a series $\sum a_k$, suppose $\lim_{k\to\infty} \sqrt[k]{|a_k|} = r$

1. If $r < 1$ the series converges absolutely
2. If $r > 1$ the series diverges
3. If $r = 1$ test is inconclusive.

Why it works: Suppose $\lim_{k\to\infty} \sqrt[k]{|a_k|} = r$

Then $\sqrt[k]{|a_k|} \approx r$ for large k.

Hence $|a_k| \approx r^k$ so $\sum |a_k| \approx \sum r^k$

acts like a geometric series with radius r.

Example $\quad \displaystyle\sum_{k=1}^{\infty} \left(\frac{1+e^k}{2e^k-1} \right)^k$

$$\lim_{k \to \infty} \sqrt[k]{|a_k|} = \lim_{k \to \infty} \sqrt[k]{\left(\frac{1+e^k}{2e^k-1} \right)^k}$$

$$= \lim_{k \to \infty} \frac{1+e^k}{2e^k-1} = \lim_{k \to \infty} \frac{e^k}{2e^k} = \frac{1}{2} < 0$$

$\boxed{\text{Series converges!}}$