CHAPTER 43

More on the Fundamental Theorem

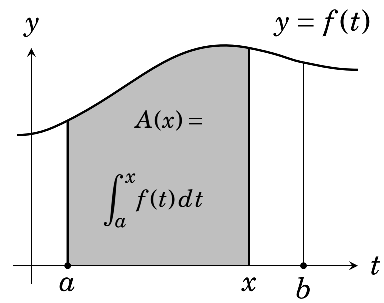
Although Part 2 of the fundamental theorem of calculus gets the greater part of our attention, Part 1 is important too, and this chapter begins with a deeper examination of it. The second part of the chapter examines a useful interpretation of Part 2.

43.1 Differentiating Integral Functions

Part 1 of the fundamental theorem of calculus states that if f is continuous on $[a,b]$, then the function $\int_a^x f(t)\,dt$ is differentiable on (a,b), and

$$D_x\left[\int_a^x f(t)\,dt\right] = f(x).$$

Recall that if $f(t) \ge 0$ on $[a,b]$, then the function $A(x) = \int_a^x f(t)\,dt$ gives the area of the region under the graph of $y = f(t)$ between $t = a$ and $t = x$. Part 2 of the FTC says that the derivative of this area function is $f(x)$. That is, *area under the graph of f is an antiderivative of f.*



This section looks at further examples of Part 1 of the FTC. Part 1 is almost deceptively simple. It says $D_x\left[\int_a^x f(t)\,dt\right] = f(x)$, that is, the derivative of $\int_a^x f(t)\,dt$ is just the integrand $f(t)$ with the variable t replaced with x.

Example 43.1 Find the derivative of $\displaystyle\int_0^x t\cos(t)\,dt$.

Solution By FTC Part 1, the answer is $D_x\left[\int_0^x t\cos(t)\,dt\right] = \boxed{x\cos(x).}$ ✍

Example 43.2 $$D_x\left[\int_0^x \frac{\ln(1+t^6)\cos(t)}{t^{11}+e^t}\,dt\right] = \boxed{\frac{\ln(1+x^6)\cos(x)}{x^{11}+e^x}.}$$ ✍

In the statement of FTC Part 1, the upper limit of integration is the variable x. When this is not the case, then you may have to switch the limits of integration (thus negating the integral) before applying FTC Part 1:

Example 43.3 $D_x\left[\int_x^2 \frac{t\sin(t)}{e^t}\,dt\right] = D_x\left[-\int_2^x \frac{t\sin(t)}{e^t}\,dt\right] = \boxed{-\frac{x\sin(x)}{e^x}}$ ✍

Example 43.4 Find $D_x\left[\int_0^{x^2+x} t\cos(t)\,dt\right]$.

Solution Notice $y = \int_0^{x^2+x} t\cos(t)\,dt$ is a composition: $\begin{cases} y = \int_0^u t\cos(t)\,dt \\ u = x^2+x. \end{cases}$
Therefore we need the chain rule to find its derivative. By the chain rule, $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$. By the FTC Part 2, $\frac{dy}{du} = \frac{d}{du}\left[\int_0^u t\cos(t)\,dt\right] = u\cos(u)$. Therefore $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = u\cos(u)(2x+1) = \boxed{(x^2+x)\cos(x^2+x)(2x+1)}$ ✍

43.2 The Definite Integral as Net Change

We now briefly explore yet another interpretation of the definite integral. The FTC says that if F is an antiderivative of f (that is, if $F' = f$), then

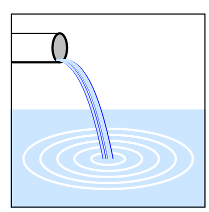
$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Because $F' = f$, this can be written as

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

On the left, the derivative $F'(x)$ is the rate of change of $F(x)$. On the right, the difference $F(b)-F(a)$ is the net change in $F(x)$ between $x = a$ and $x = b$.

$$\int_a^b \left(\begin{array}{c}\text{rate of change}\\ \text{of quantity } F(x)\end{array}\right)dx = F(b) - F(a) = \left(\begin{array}{c}\text{net change in } F(x)\\ \text{between } a \text{ and } b\end{array}\right).$$



For example, suppose $r(t)$ is the rate of change (in liters/minute) of water pouring from a pipe at time t. So $r(t)$ is the rate of change in the quantity (liters) of water that has poured out of the pipe at time t. The integral $\int_5^{10} r(t)\,dt$ is the number of liters that poured out between times 5 and 10.

For another example, suppose an object moving on a line has velocity $v(t)$ at time t. Since $v(t)$ is the rate of change of its position $s(t)$, a definite integral like $\int_a^b v(t)\,dt = s(b) - s(a)$ is the net change in distance between times a and b. The net change in distance is *not necessarily* how far the object has traveled between times a and b, but rather how far it is at time b

from where it was at time a. For example, if at time b it has returned to where is was at time a, then $\int_a^b v(t)\,dt = s(b) - s(a) = 0$.

Exercises for Chapter 43

1. Differentiate $\displaystyle\int_0^x \cos(t)\,dt$ in two ways. First, use FTC Part 1. Second, use FTC Part 2 to find the integral, and differentiate the resulting function.

2. Differentiate $\displaystyle\int_2^x \left(t^5 + e^t + 3\right)dt$ in two ways. First, use FTC Part 1. Second, use FTC Part 2 to find the integral, and differentiate the resulting function.

3. Differentiate $\displaystyle\int_1^x \left(\sec^2(t) + \sec(t)\tan(t)\right)dt$ in two ways. First, use FTC Part 1. Second, use FTC Part 2 to find the integral, and differentiate the resulting function.

4. Differentiate $\displaystyle\int_0^x \frac{1}{\sqrt{1-t^2}}\,dt$ in two ways. First, use FTC Part 1. Second, use FTC Part 2 to find the integral, and differentiate the resulting function.

5. Differentiate $\displaystyle\int_0^x t^3\cos(t)e^t\,dt$. Then differentiate $\displaystyle\int_x^0 t^3\cos(t)e^t\,dt$.

6. Differentiate $\displaystyle\int_0^x \sqrt{10+\sin(t)+\cos(t)}\,dt$. Then differentiate $\displaystyle\int_x^0 \sqrt{10+\sin(t)+\cos(t)}\,dt$.

7. Differentiate: $\displaystyle y = \int_0^{\sqrt{x}+1} t^3\cos(t)e^t\,dt$

8. Differentiate: $\displaystyle f(x) = \int_1^{x^2-x+2} \cos\left(\frac{t}{t+1}\right)dt$

9. Differentiate: $\displaystyle f(x) = \int_\pi^{\sin(x)} e^{t^3+2t}\,dt$

10. Differentiate: $\displaystyle f(x) = \int_\pi^{\sqrt{x}+1} \frac{t^2-1}{t^2+1}\,dt$

11. Differentiate: $\displaystyle f(x) = \int_{5x^2}^{8} \sqrt{t^5-7}\,dt$

12. Differentiate: $\displaystyle f(z) = \int_{z^4+z^2}^{10} \sqrt{\frac{t}{t^5-7}}\,dt$

13. Differentiate: $\displaystyle y = \int_x^{x^2-3x} \sin\left(t^2\right)\sqrt[3]{t}\,dt$

14. Differentiate: $\displaystyle f(x) = \int_{-x}^{x} \sin\left(t^2-2\right)t^2\,dt$

15. Suppose $\displaystyle F(x) = \int_2^x \frac{t-3}{t^2+7}\,dt$. Find $F'(x)$ and $F''(x)$. Find $F(2)$. Find where F attains its global minimum. Find where F is concave up/down. Use this information to sketch the graph of F.

16. Find the equation of the tangent line to the graph of $\displaystyle f(x) = \int_{-2}^{x} \frac{t^3}{\sqrt{t^2+5}}\,dt$ at the point $\left(-2, f(-2)\right)$.

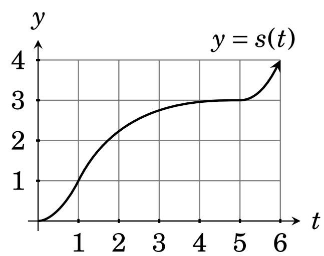
17. For what x is the tangent to $\displaystyle y = \int_2^{x^2+1} \sqrt[3]{t^2+3t+2}\,dt$ horizontal?

18. Suppose an object moving on a line has a velocity of $v(t) = 3t^2 - 2t + 1$ units per second at time t seconds. What is the difference in its positions between times $t = 1$ and $t = 4$?

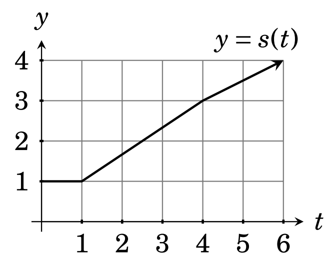
19. Suppose $r(t)$ is the rate, in acres per day, of US farmland being lost to development t days after January 1, 2022. Suppose $\int_{32}^{60} r(t)\,dt = 3127$. What does this mean?

20. Suppose $f(t)$ is the rate, in tons per day, of trash New York City exports to North Carolina landfills t days after January 1, 2021. Suppose $\int_{0}^{31} r(t)\,dt = 375{,}032$. What does this mean?

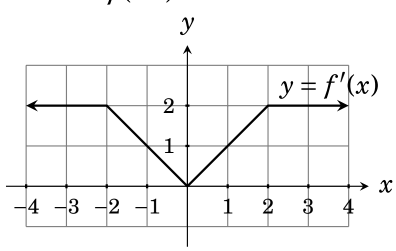
21. An object moving on a line has position $s(t)$ and velocity $v(t)$ at time t. The position function $s(t)$ is graphed below. Find $\int_1^5 v(t)\,dt$.



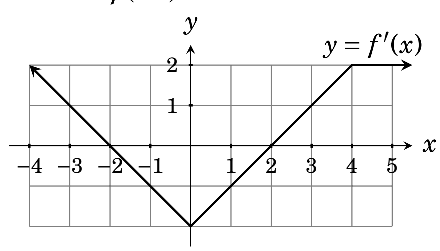
22. An object moving on a line has position $s(t)$ and velocity $v(t)$ at time t. The position function $s(t)$ is graphed below. Find $\int_0^4 v(t)\,dt$.



23. The derivative $f'(x)$ of a function $f(x)$ is graphed below. If $f(2) = 3$, what is $f(-3)$?



24. The derivative $f'(x)$ of a function $f(x)$ is graphed below. If $f(5) = 3$, what is $f(-2)$?



Exercise Solutions for Chapter 43

1. Differentiate $\int_0^x \cos(t)\,dt$ in two ways.

By FTC Part 1, $D_x\left[\int_0^x \cos(t)\,dt\right] = \cos(x)$. Using FTC Part 2, $D_x\left[\int_0^x \cos(t)\,dt\right] = D_x\Big[\,\big[\sin(t)\big]_0^x\,\Big] = D_x\Big[\sin(x)-\sin(0)\Big] = D_x\Big[\sin(x)\Big] = \cos(x)$.

3. Differentiate $\int_1^x \left(\sec^2(t)+\sec(t)\tan(t)\right)dt$ in two ways.

By FTC Part 1, $D_x\left[\int_0^x \left(\sec^2(t)+\sec(t)\tan(t)\right)dt\right] = \sec^2(x)+\sec(x)\tan(x)$.

Using FTC Part 2, $D_x\left[\int_0^x \left(\sec^2(t)+\sec(t)\tan(t)\right)dt\right] = D_x\Big[\big[\tan(t)+\sec(t)\big]_0^x\Big] =$

$D_x\Big[(\tan(x)+\sec(x))-(\tan(0)+\sec(0))\Big] = D_x\Big[\tan(x)+\sec(x)\Big] = \sec^2(x)+\sec(x)\tan(x)$

5. Differentiate $\int_0^x t^3\cos(t)e^t\,dt$. Then differentiate $\int_x^0 t^3\cos(t)e^t\,dt$.

$D_x\Big[\int_0^x t^3\cos(t)e^t\,dt\Big] = x^3\cos(x)e^x$

$D_x\Big[\int_x^0 t^3\cos(t)e^t\,dt\Big] = D_x\Big[-\int_0^x t^3\cos(t)e^t\,dt\Big] = -x^3\cos(x)e^x$

7. Differentiate the function $f(x) = \int_0^{\sqrt{x}+1} t^3\cos(t)e^t\,dt$.

Write this as a composition $y = \int_0^u t^3\cos(t)e^t\,dt$ and $u = \sqrt{x}+1$. By the chain rule, $f'(x) = \dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = u^2\cos(u)e^u\cdot\dfrac{1}{2\sqrt{x}} = \dfrac{\left(\sqrt{x}+1\right)^2\cos\left(\sqrt{x}+1\right)e^{\sqrt{x}+1}}{2\sqrt{x}}$.

9. Differentiate the function $f(x) = \int_\pi^{\sin(x)} e^{t^3+2t}\,dt$.

Write this as a composition $y = \int_\pi^u e^{t^3+2t}\,dt$ and $u = \sin(x)$. By the chain rule, $f'(x) = \dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = e^{u^3+2u}\cos(x) = e^{\sin^3(x)+2\sin(x)}\cos(x)$.

11. Differentiate: $f(x) = \int_{5x^2}^8 \sqrt{t^5-7}\,dt$

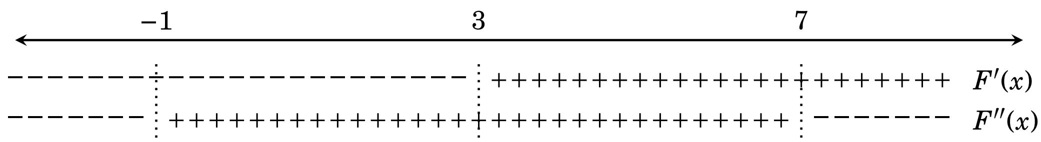
First, put this into the form $f(x) = -\int_8^{5x^2}\sqrt{t^5-7}\,dt$, so that FTC 1 applies. Now $f(x)$ is a composition of $y = -\int_8^u \sqrt{t^5-7}\,dt$ with $u = 5x^2$. By the chain rule, $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = -\sqrt{u^5-7}\;10x = -\sqrt{(5x^2)^5-7}\;10x = -\sqrt{3125x^{10}-7}\;10x$.

13. Differentiate: $y = \int_x^{x^2-3x} \sin\left(t^2\right)\sqrt[3]{t}\,dt$

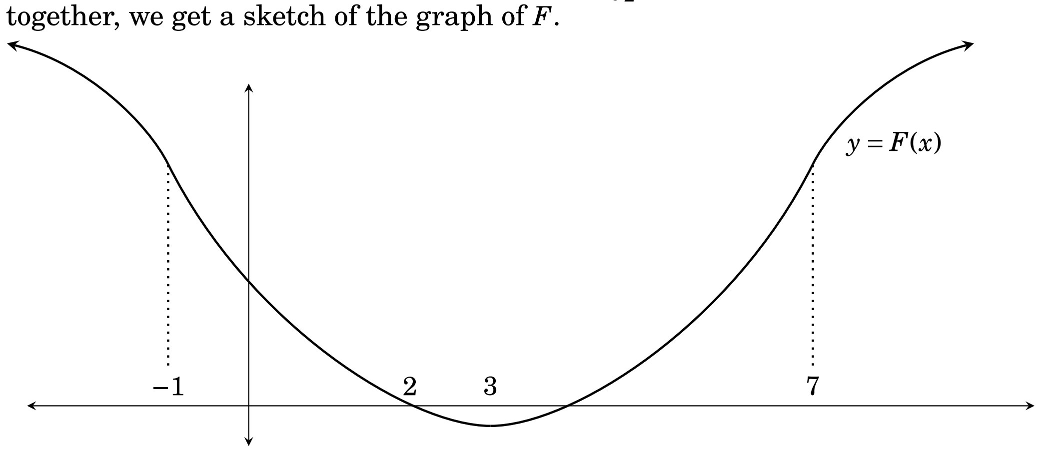
To begin, break this up as $y = \int_x^0 \sin(t^2)\sqrt[3]{t}\,dt + \int_0^{x^2-3x} \sin(t^2)\sqrt[3]{t}\,dt$, which yields $y = -\int_0^x \sin(t^2)\sqrt[3]{t}\,dt + \int_0^{x^2-3x} \sin(t^2)\sqrt[3]{t}\,dt$. Using FTC 1 and the chain rule, we get $y' = -\sin(x^2)\sqrt[3]{x} + \sin\left((x^2-3x)^2\right)\sqrt[3]{x^2-3x}\,(2x-3)$.

15. Suppose $F(x) = \int_2^x \frac{t-3}{t^2+7}\,dt$. Find $F'(x)$ and $F''(x)$. Find $F(2)$. Find where F attains its global minimum. Find where F is concave up/down. Use this information to sketch the graph of F.

By FTC part 1, $F'(x) = \frac{x-3}{x^2+7}$. By the quotient rule, $F''(x) = \frac{1\cdot(x^2+7)-(x-3)2x}{\left(x^2+7\right)^2} = \frac{-x^2+6x+7}{\left(x^2+7\right)^2} = \frac{-(x+1)(x-7)}{\left(x^2+7\right)^2}$. Looking at the factored forms of F' and F'', we can read off the signs of these derivatives as follows.



From this we can see that F decreases on $(-\infty,3)$ and increases on $(3,\infty)$. Therefore F has a global minimum at $x=3$. Also F is concave up on $(-1,7)$ and concave down on $(-\infty,-1)\cup(7,\infty)$. In addition, $F(2) = \int_2^2 \frac{t-3}{t^2+7}\,dt = 0$. Putting all this together, we get a sketch of the graph of F.



17. For what x is the tangent to $y = \int_2^{x^2+1} \sqrt[3]{t^2+3t+2}\,dt$ horizontal?

This function is a composition $y = \int_2^u \sqrt[3]{t^2+3t+2}\,dt$ with $u = x^2+1$. By the chain rule, $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \sqrt[3]{u^2+3u+2}\cdot 2x = \sqrt[3]{(x^2+1)^2+3(x^2+1)+2}\cdot 2x$. This certainly equals 0 if $x=0$. Can other values of x make the derivative zero? Because x^2+1 is positive for any x, it follows that $\sqrt[3]{(x^2+1)^2+3(x^2+1)+2} > 0$. Therefore $\frac{dy}{dx}$ is zero if and only if $x=0$.

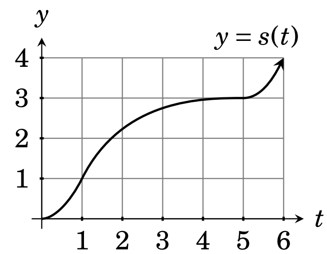
Answer The tangent line is horizontal only at $x=0$.

19. Suppose $r(t)$ is the rate, in acres per day, of US farmland being lost to development t days after January 1, 2022. Suppose $\int_{32}^{60} r(t)\,dt = 3127$. What does this mean?

Answer Between the 32nd and 60th day of 2022 (i.e., in the month of February), 3127 acres of farmland were lost to development.

21. An object moving on a line has position $s(t)$ and velocity $v(t)$ at time t. The position function $s(t)$ is graphed below. Find $\int_1^5 v(t)\,dt$.

$$\int_1^5 v(t)\,dt = \Big[s(t)\Big]_1^5 = s(5)-s(1) = 3-1 = 2$$



23. The derivative $f'(x)$ of a function $f(x)$ is graphed below. If $f(2)=3$, what is $f(-3)$?

By FTC 2, $f(2)-f(-3) = \Big[f(x)\Big]_{-3}^{2} = \int_{-3}^{2} f'(x)\,dx$.

By area under graph, $\int_{-3}^{2} f'(x)\,dx = 6$.

So $f(2)-f(-3) = 6$. Hence $f(-3) = f(2)-6 = 3-6 = -3$.