CHAPTER 38

Antiderivatives

We are at a major turning point of the course. Up until now our primary focus has been on the process of *differentiation*. Given a function $f(x)$, find its derivative $f'(x)$.

$$f(x) \xrightarrow{\text{differentiation}} f'(x)$$

Starting now, we shift our focus to the reverse process. If you know the derivative $f'(x)$, can you find the function $f(x)$? This reverse process is called **antidifferentiation** or **integration.**

$$f(x) \xleftarrow[\text{antidifferentiation}]{\text{integration}} f'(x)$$

With this goal in mind, we state the chapter's main definition.

Definition 38.1
A function $F(x)$ is an **antiderivative** of $f(x)$ if $D_x\Big[F(x)\Big] = f(x)$.

$$F(x) \xrightarrow{\text{differentiate}} f(x)$$

antiderivative of $f(x)$ — derivative of $F(x)$

For example, let's find an antiderivative of the function $f(x) = 2x$. We ask: what function could we differentiate that would produce a derivative of $2x$?

$$? \xrightarrow{\text{differentiate}} 2x$$

In essence we are asking $D_x[?] = 2x$. Because $D_x[x^2] = 2x$, the function x^2 is an antiderivative of $2x$.

Actually, there are *lots* of functions whose derivatives are $2x$:

$$\begin{aligned} D_x[x^2] &= 2x \\ D_x[x^2+1] &= 2x \\ D_x[x^2-2] &= 2x \\ D_x[x^2+\pi] &= 2x \end{aligned}$$

In general, if C is any constant whatsoever,

$$D_x[x^2+C] = 2x.$$

Thus the function $f(x)=2x$ has infinitely many antiderivatives $F(x)=x^2+C$. Their graphs are the graph of $y=x^2$ raised (or lowed) by C units.

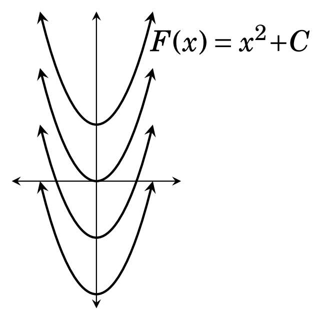

Figure 38.1. The antiderivatives of the function $f(x)=2x$.

Likewise, a little reverse engineering tells us that the antiderivatives of the function $f(x)=3x^2$ are the functions $F(x)=x^3+C$ (where C is a constant) because $D_x[x^2+C]=3x^2$. Their graphs are indicated below.

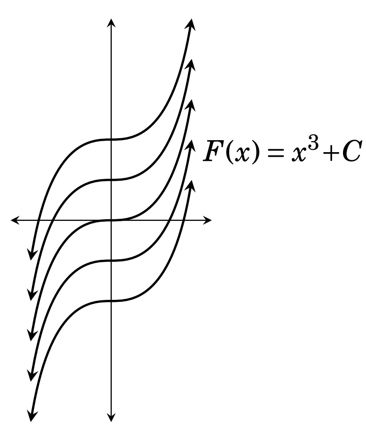

Figure 38.2. The antiderivatives of the function $f(x)=3x^2$.

So if $f(x)$ is a function that has an antiderivative $F(x)$, then $f(x)$ has not one but infinitely many antiderivatives $F(x)+C$, where C is any constant. There is a special name and notation for these antiderivatives.

Definition 38.2 Suppose f is a continuous function on an interval. The set of all antiderivatives of f is called the **indefinite integral** of f. This set of functions is denoted by

$$\int f(x)\,dx.$$

Thus $\int f(x)dx$ stands for the set of all functions whose derivative is $f(x)$. We typically write $\int f(x)dx = F(x)+C$ where C denotes a constant and $D_x\big[F(x)+C\big] = f(x)$. We read $\int f(x)dx$ as "*the indefinite integral of $f(x)$ dx*" or just "*the integral of $f(x)$ dx*".

Example: $\displaystyle\int 2x\,dx = x^2+C$, where C is a constant.

Example: $\displaystyle\int 3x^2\,dx = x^3+C$, where C is a constant.

Given $\int f(x)dx$, the process of finding $F(x)+C$ is called **integration**. In the examples above we found $F(x)+C$ simply from our experience with differentiation, but we will shortly develop a set of integration formulas.

In the expression $\int f(x)dx$, the symbol $\int$ is called the **integral sign** and the function $f(x)$ (the function being integrated) is called the **integrand**. The dx is called a *differential*. We will have more to say about differentials in Chapter 39, but for now it's best to think of the dx as punctuation, like a closing parenthesis.

Remember that $\displaystyle\int f(x)\,dx = F(x)+C$ means that $D_x\Big[F(x)+C\Big] = f(x)$. This is so important that we will display it in a box and revisit it many times.

$$\boxed{\int f(x)dx \;=\; F(x)+C \quad\Longleftrightarrow\quad D_x\Big[F(x)+C\Big] = f(x)}$$

Example: $\displaystyle\int x^3\,dx \;=\; \frac{1}{4}x^4+C$ because $\displaystyle D_x\left[\frac{1}{4}x^4+C\right] = x^3.$

Example: $\displaystyle\int \frac{2x}{1+x^2}\,dx \;=\; \ln\big(1+x^2\big)+C$ because $\displaystyle D_x\Big[\ln\big(1+x^2\big)+C\Big] = \frac{2x}{1+x^2}.$

Example: $\displaystyle\int \frac{1}{1+x^2}\,dx \;=\; \tan^{-1}(x)+C$ because $\displaystyle D_x\Big[\tan^{-1}(x)+C\Big] = \frac{1}{1+x^2}.$

These examples were somewhat ad hoc. But they underscore the fact that we need a systematic set of rules for finding indefinite integrals. We will begin that task now. Fortunately the task is relatively easy, because for every derivative rule we get a corresponding integral rule by "running the derivative rule in reverse." We will start with the power rule.

For powers $n \neq -1$ we get an immediate rule for $\int x^n\,dx$ as follows.

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}+C \text{ because } D_x\left[\frac{1}{n+1}x^{n+1}+C\right] = \frac{1}{n+1}(n+1)x^n = x^n.$$

This is called the *power rule for integration*.

Power Rule for Integration: $\displaystyle\int x^n\,dx = \frac{1}{n+1}x^{n+1}+C \quad (\text{provided } n \neq -1)$

Example: $\displaystyle\int x^3\,dx = \frac{1}{3+1}x^{3+1}+C = \frac{1}{4}x^4+C$ ✍

Example: $\displaystyle\int x^8\,dx = \frac{1}{8+1}x^{8+1}+C = \frac{1}{9}x^9+C$ ✍

Example: $\displaystyle\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{1}{1/2+1}x^{1/2+1}+C = \frac{2}{3}x^{3/2}+C = \frac{3}{2}\sqrt{x}^3+C$ ✍

Example: $\displaystyle\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx = \frac{1}{-3+1}x^{-3+1}+C = -\frac{1}{2}x^{-2}+C = \frac{1}{2x^2}+C$ ✍

The power rule for integration breaks down for $n=-1$ because it would read $\int x^{-1}dx = \frac{1}{-1+1}x^{-1+1}+C$, and this involves division by zero.

So is there a formula for $\int x^{-1}dx$? That is, is there a formula for $\int \frac{1}{x}dx$? The answer would have to be a function $F(x)+C$ whose derivative is $\frac{1}{x}$. We don't have to look far: Because $D_x\left[\ln|x|+C\right] = \frac{1}{x}$, we have our next rule.

Power Rule for n = −1: $\displaystyle\int \frac{1}{x}\,dx = \ln|x|+C$

What other easy integration rules are within reach? If c is a constant (possibly different from C), then $D_x\left[cx+C\right] = c$. This yields another rule.

Constant Rule for Integration: $\displaystyle\int c\,dx = cx+C$

Example: $\displaystyle\int 5\,dx = 5x+C$

Example: $\displaystyle\int \sqrt{2}\,dx = \sqrt{2}x+C$

Any derivative rule $D_x[F(x)] = f(x)$ yields an integration rule $\int f(x)\,dx = F(x) + C$. For example, from $D_x[\sin(x)] = \cos(x)$ we get $\int \cos(x)\,dx = \sin(x) + C$. From $D_x[\cos(x)] = -\sin(x)$ we get $\int \sin(x)\,dx = -\cos(x) + C$.

Here is a list, beginning with the three formulas from the previous page. Each rule below has the form $\int f(x)\,dx = F(x) + C$. Check that each rule is correct by verifying $D_x[F(x) + C] = f(x)$.

Integration Rules

$$\begin{aligned}
\int c\,dx &= cx + C \\
\int x^n\,dx = &= \frac{1}{n+1}x^{n+1} + C \quad (\text{if } n \neq -1) \\
\int x^{-1}\,dx &= \ln|x| + C \\
\int e^x\,dx &= e^x + C \\
\int b^x\,dx &= \frac{1}{\ln(b)}b^x + C \\
\int \sin(x)\,dx &= -\cos(x) + C \\
\int \cos(x)\,dx &= \sin(x) + C \\
\int \sec^2(x)\,dx &= \tan(x) + C \\
\int \csc^2(x)\,dx &= -\cot(x) + C \\
\int \sec(x)\tan(x)\,dx &= \sec(x) + C \\
\int \csc(x)\cot(x)\,dx &= -\csc(x) + C \\
\int \frac{1}{\sqrt{1-x^2}}\,dx &= \sin^{-1}(x) + C \\
\int \frac{1}{1+x^2}\,dx &= \tan^{-1}(x) + C \\
\int \frac{1}{x\sqrt{x^2-1}}\,dx &= \sec^{-1}|x| + C
\end{aligned}$$

These integration rules, or “backwards derivative rules” are easy to internalize and memorize because of our experience with derivative rules.

But notice that there are certain glaring omissions. For instance, there is no formula for $\int \tan(x)\,dx$ because we don't have a derivative rule of form $D_x\Big[F(x)\Big] = \tan(x)$. Formulas for $\int \tan(x)\,dx$, $\int \cot(x)\,dx$, $\int \sec(x)\,dx$, $\int \ln(x)\,dx$, etc., will have to wait until Calculus II.

Never forget that $\int f(x)\,dx = F(x)+C$ means $D_x\big[F(x)+C\big] = f(x)$. So if you integrate $f(x)$, then differentiate the result, you get $f(x)$ back. In symbols,

$$D_x\left[\int f(x)\,dx\right] = f(x).$$

One consequence of this is another integration rule.

Constant Multiple Rule for Integration: If c is a constant, then

$$\int c f(x)\,dx = c\int f(x)\,dx.$$

To check that this is correct, we can differentiate the right-hand side and see if we get the integrand $c\,f(x)$ from in the integral on the left. Doing so,

$$D_x\left[c\int f(x)\,dx\right] = c\,D_x\left[\int f(x)\,dx\right] = c\,f(x).$$

Since we got $c\,f(x)$, the formula is correct.

The constant multiple rule combines with the other integration formulas:

Example: $\displaystyle\int 7x^3 dx = 7\int x^3 dx = 7\frac{1}{3+1}x^{3+1}+C = \frac{7}{4}x^4+C$ ✍

Example: $\displaystyle\int \frac{\pi}{x}dx = \pi\int \frac{1}{x}dx = \pi\ln|x|+C$ ✍

Another integration rule comes from reversing the sum-difference rule for derivatives. Check it by differentiating the right side to get $f(x)\pm g(x)$:

Sum-Difference Rule for Integration:

$$\int \big(f(x)\pm g(x)\big)\,dx = \int f(x)\,dx \pm \int g(x)\,dx.$$

A great many indefinite integrals can be found by combining the rules on this page with those on previous page. For example:

$$\begin{aligned}
\int \big(5x^2+2x\big)\,dx &= \int 5x^2\,dx + \int 2x\,dx && \text{(sum-difference rule)}\\
&= 5\int x^2\,dx + 2\int x^1\,dx && \text{(const. mult. rule, twice)}\\
&= 5\frac{1}{3}x^3 + 2\frac{1}{2}x^2 + C && \text{(power rule, twice)}\\
&= \frac{5}{3}x^3 + x^2 + C
\end{aligned}$$

In the third step we just added a C to the end, rather than getting two constants (one from each integral) and adding them.

We did this example in four steps, but after some practice you'll work problems like this in one step.

Example: $\displaystyle\int \big(\pi\cos(x) - 3\sec^2(x) - 3x^2 + 4\big)\,dx \;=\; \pi\sin(x) - 3\tan(x) - x^3 + 4x + C.$

Always remember that you can check your answer to an integration problem by differentiating your answer and seeing if that produces the integrand. In this example, $D_x\big[\pi\sin(x) - 3\tan(x) - x^3 + 4x + C\big] \;=\; \pi\cos(x) - 3\sec^2(x) - 3x^2 + 4$. That is the integrand, so we know we integrated correctly.

Work enough exercises that you can do such problems readily.

Our examples here and the exercises below use the variable x exclusively. But any variable can be used.

Example: $\displaystyle\int \big(u + 3e^u\big)\,du \;=\; \frac{1}{2}u^2 + 3e^u + C$. Don't forget that the differential du must match the variable u (use du here, and not dx).

The next example illustrates that occasionally some algebraic manipulation is needed to bring a problem to a form that matches a rule.

Example: Find $\displaystyle\int \big(w^2 - 3w\big)(w+1)\,dw$.

This does not match any integration rules, but we can put it into a manageable form by multiplying the binomials before integrating.

$$\int \big(w^2 - 3w\big)(w+1)\,dw \;=\; \int \big(w^3 - 2w^2 - 3w\big)\,dw \;=\; \frac{w^4}{4} - \frac{2w^3}{3} - \frac{3w^2}{2} + C$$

There is no product rule for integration, that is, no rule for $\displaystyle\int f(x)g(x)\,dx$. Reversing the product rule for derivatives would yield

$$\int \big(f'(x)g(x) + f(x)g'(x)\big)\,dx \;=\; f(x)g(x) + C.$$

This is not very useful, as few functions match the form $f'(x)g(x) + f(x)g'(x)$. But in Chapter **??** we will see that reversing the *chain rule* is very useful.

In conclusion, we have begun exploring integration, the opposite process of differentiation. This theme will occupy us for the remainder of the course.

$$\begin{tikzcd}[column sep=huge]
F(x)+C \arrow[r, bend left, "\text{differentiation}", "D_x"'] & f(x) \arrow[l, bend left, "\text{integration}", "\int dx"']
\end{tikzcd}$$

Exercises for Chapter 38

Find the indicated indefinite integrals.

1. $\int \left(5x+3+x^4\right)\,dx$

2. $\int \left(x^5+x+1\right)\,dx$

3. $\int \left(4x^5+x+2\right)\,dx$

4. $\int \left(x^3+3x+5\right)\,dx$

5. $\int \left(5x^2+2+\sin(x)\right)\,dx$

6. $\int \left(2e^x+x^4+\sec(x)\tan(x)\right)\,dx$

7. $\int \left(7+x^6+\sec^2(x)\right)\,dx$

8. $\int \left(3x^2+\sin(x)+3\right)\,dx$

9. $\int \left(e^x+e+\csc^2(x)\right)\,dx$

10. $\int 5x^{-1}\,dx$

11. $\int 3\sec(x)\tan(x)\,dx$

12. $\int \left(4x^3+\cos(x)+1\right)dx$

13. $\int \left(\sqrt[3]{x}+\cos(x)\right)\,dx$

14. $\int \sqrt[5]{x}^3\,dx$

15. $\int \left(e^x+x^4+3\right)\,dx$

16. $\int \left(\sec^2(x)+3\sin(x)\right)\,dx$

17. $\int \left(x^3+2x+e^x\right)\,dx$

18. $\int 6\sqrt{x}\,dx$

19. $\int \left(4x+\frac{1}{x}+\sin(x)\right)\,dx$

20. $\int \left(\frac{1}{x^3}+\sqrt{x}\right)\,dx$

21. $\int \frac{5}{1+x^2}\,dx$

22. $\int \left(\frac{1}{x}+\cos(x)\right)\,dx$

23. $\int \frac{2}{x\sqrt{x^2-1}}\,dx$

24. $\int \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)\,dx$

25. $\int \frac{\pi}{\sqrt{1-x^2}}\,dx$

26. $\int \left(\sqrt[3]{x}+\frac{1}{x^4}\right)\,dx$

27. $\int \frac{\pi}{3+3x^2}\,dx$

28. $\int x+\frac{1}{\sqrt{1-x^2}}\,dx$

29. $\int \frac{1}{\sqrt{x}^5}\,dx$

30. $\int \left(x^2+\frac{1}{x^2}+e\right)\,dx$

31. $\int \frac{e^{2x}+e^x}{e^x}\,dx$

32. $\int \left(x^4+\frac{1}{x}+\sqrt{2}\right)\,dx$

33. $\int \frac{1}{x^2}\,dx$

34. $\int \frac{x^2+1}{2x}\,dx$

35. $\int \frac{x^3-3x^2+1}{x^2}\,dx$

36. $\int \left(x^2+1\right)(2x+1)\,dx$

37. Is the equation $\int x\cos(x)\,dx = x\sin(x)+\cos(x)+C$ true or false?

38. Is the equation $\int \left(\cos(x)\frac{1}{x}-\sin(x)\ln(x)\right)dx = \cos(x)\ln(x)+C$ true or false?

39. Is the equation $\int \frac{\sin(1/x)}{x^2}dx = \cos\left(\frac{1}{x}\right)+C$ true or false?

40. Is the equation $\int x\cos(x)\,dx = \frac{x^2}{2}\sin(x)+C$ true or false?

41. If $f(x)$ and $g(x)$ are differentiable functions, find $\int \left(f'(x)g(x)+f(x)g'(x)\right)dx$.

42. If $f(x)$ and $g(x)$ are differentiable functions, find $\int f'\big(g(x)\big)g'(x)\,dx$.

43. If $f(x)$ and $g(x)$ are differentiable functions, find $\int \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}\,dx$.

Exercise Solutions for Chapter 38

1. $\int \left(5x+3+x^4\right)\,dx = \frac{5x^2}{2}+3x+\frac{x^5}{5}+C$

3. $\int \left(4x^5+x+2\right)\,dx = \frac{2x^6}{3}+\frac{x^2}{2}+2x+C$

5. $\int \left(5x^2+2+\sin(x)\right)\,dx = \frac{5x^3}{3}+2x-\cos(x)+C$

7. $\int \left(7+x^6+\sec^2(x)\right)\,dx = 7x+\frac{x^7}{7}+\tan(x)+C$

9. $\int \left(e^x+e+\csc^2(x)\right)\,dx = e^x+ex-\cot(x)+C$

11. $\int 3\sec(x)\tan(x)\,dx = 3\sec(x)+C$

13. $\int \left(\sqrt[3]{x}+\cos(x)\right)\,dx = \int \left(x^{1/3}+\cos(x)\right)\,dx = \frac{1}{1/3+1}x^{1/3+1}+\sin(x)+C = \frac{1}{4/3}x^{4/3}+\sin(x)+C$
$= \frac{3}{4}\sqrt[3]{x}^4+\sin(x)+C$

15. $\int \left(e^x+x^4+3\right)\,dx = e^x+\frac{x^5}{5}+3x+C$

17. $\int \left(x^3+2x+e^x\right)\,dx = \frac{x^4}{4}+x^2+e^x+C$

19. $\int \left(4x+\frac{1}{x}+\sin(x)\right)\,dx = 2x^2+\ln|x|-\cos(x)+C$

21. $\int \frac{5}{1+x^2}\,dx = 5\int \frac{1}{1+x^2} = 5\tan^{-1}(x)+C$

23. $\int \frac{2}{x\sqrt{x^2-1}}\,dx = 2\int \frac{1}{x\sqrt{x^2-1}}\,dx = 2\sec^{-1}|x|+C$

25. $\int \frac{\pi}{\sqrt{1-x^2}}\,dx = \pi\int \frac{1}{\sqrt{1-x^2}}\,dx = \pi\sin^{-1}(x)+C$

27. $\int \frac{\pi}{3+3x^2}\,dx = \frac{\pi}{3}\int \frac{1}{1+x^2}\,dx = \frac{\pi}{3}\tan^{-1}(x)+C$

29. $\int \frac{1}{\sqrt{x}^5}\,dx = \int x^{-5/2}dx = \frac{1}{-5/2+1}x^{-5/2+1}+C = \frac{1}{-3/2}x^{-3/2}+C = -\frac{2}{3\sqrt{x}^3}+C$

31. $\int \frac{e^{2x}+e^x}{e^x}\,dx = \int \frac{e^{2x}}{e^x}+\frac{e^x}{e^x}\,dx \int \left(e^x+1\right)\,dx = e^x+x+C$

33. $\int \frac{1}{x^2}\,dx = \int x^{-2}dx = \frac{1}{-2+1}x^{-2+1} + C = -x^{-1} + C = -\frac{1}{x} + C$

35. $\int \frac{x^3 - 3x^2 + 1}{x^2}\,dx = \int \left(x - 3 + x^{-2}\right)dx = \frac{x^2}{2} - 3x - \frac{1}{x} + C$

37. Is the equation $\int x\cos(x)\,dx = x\sin(x) + \cos(x) + C$ true or false?

Since $D_x\left[x\sin(x) + \cos(x) + C\right] = \sin(x) + x\cos(x) - \sin(x) = x\cos(x)$, this is true.

39. Is the equation $\int \frac{\sin\left(\frac{1}{x}\right)}{x^2}dx = \cos\left(\frac{1}{x}\right) + C$ true or false?

Since $D_x\left[\cos\left(\frac{1}{x}\right) + C\right] = -\sin\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) = \frac{\sin\left(\frac{1}{x}\right)}{x^2}$ equals the integrand, this is **true**.

41. $\int \left(f'(x)g(x) + f(x)g'(x)\right)dx = f(x)g(x) + C$ because $D_x\left[f(x)g(x) + C\right] = f'(x)g(x) + f(x)g'(x)$.

43. $\int \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}\,dx = \frac{f(x)}{g(x)} + C$ because $D_x\left[\frac{f(x)}{g(x)} + C\right] = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$.