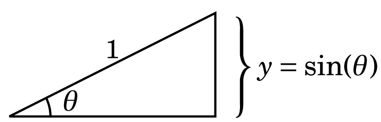
CHAPTER 3

Trigonometric Functions

Recall that a function expresses a relationship between two variable quantities. Trigonometric functions are special kinds of functions that express relationships between the *angles* of right triangles and their *sides*. For example, consider the right triangle (with hypotenuse 1) drawn below. The relationship between the side length y and the angle θ is given by the function $y = \sin(\theta)$.

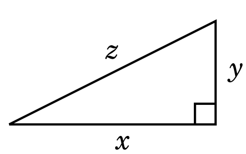


You have studied trigonometric functions before but may need a quick review to attain the fluency that this course demands. This chapter summarizes the main definitions and properties of trigonometric functions. Even if you are thoroughly familiar with this topic it is still a good idea to scan this material to glean the notation and conventions used in this text.

3.1 The Trigonometric Functions

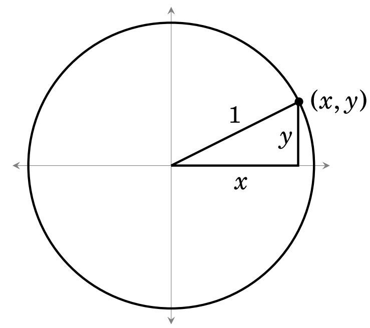
Trigonometric functions are actually very simple. Mastering them requires knowledge of only two things: The Pythagorean theorem and the unit circle.

Pythagorean Theorem: If a right triangle has legs of lengths x and y, and hypotenuse of length z, then it is always the case that $x^2 + y^2 = z^2$.



Conversely, if the sides of a triangle obey the equation $x^2 + y^2 = z^2$, then the triangle is a right triangle and the hypotenuse has length z.

The unit circle is the circle of radius 1 that is centered at the origin. It is the graph of the equation $x^2 + y^2 = 1$. That is, it is the set of all points (x, y) on the plane for which $x^2 + y^2 = 1$. To see why this is so, take any point (x, y) on the circle. It is at distance 1 from the origin. By the Pythagorean theorem, the point (x, y) satisfies $x^2 + y^2 = 1^2$.



Because it has radius 1, the unit circle has diameter 2. Its circumference, which is π times the diameter, is therefore 2π.

The unit circle is important because it is a natural protractor for measuring angles; but instead of measuring them in degrees, it measures in what are called *radians*. To understand this, say we want to measure the angle in Figure 3.1. One way to do this is to place a protractor on the angle and get a measurement, in this case 45 degrees. On the other hand, we could place the unit circle on the angle as shown on the right of Figure 3.1. Now measure the angle not by degrees, but by the arc length along the circle between the two rays of the angle. As 45° is one-eighth of the way around the circle, this arc length is one-eighth of the circumference of the circle, that is, $\frac{1}{8}2\pi = \frac{\pi}{4}$. We say that $\frac{\pi}{4}$ is the **radian measure** of the angle. In this way any angle has a radian measure, namely the arc length of the part of the unit circle that is enclosed between the angle's rays.

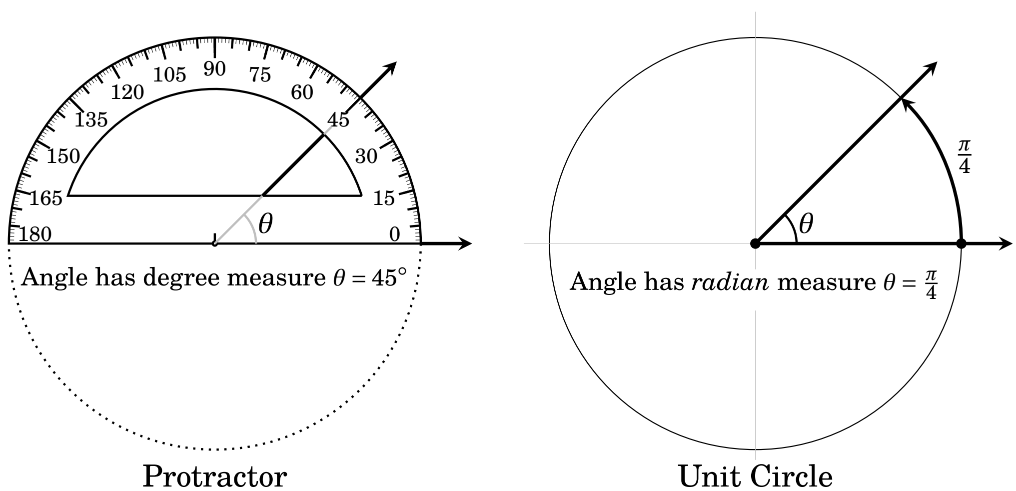


Figure 3.1. Angles can be measured with a protractor (in degrees) or with the unit circle (in radians).

Radians are considered preferable to degrees. There is a good reason for this. The protractor in Figure 3.1 is a man-made device; the fact that there are 360 degrees around circle is a mere arbitrary contrivance of the human mind. Degree measurement was arranged this way because lots of numbers go evenly into 360. By contrast, the unit circle is a universal mathematical principle. Consequently, many equations will work out neatly—and naturally—when angles are expressed in radians. For this reason we almost always use radians in calculus, even though we may sometimes think informally in degrees.

Figure 3.2 shows some angles that arise frequently in computations. The left side shows angles that are integer multiples of 45°, or $\pi/4$ radians. From this we see that 90° (twice 45°) is $\frac{\pi}{2}$ (twice $\frac{\pi}{4}$) radians. Similarly 135° is $\frac{3\pi}{2}$ radians, and 180° is π radians, etc.

If we go all around the unit circle (360°), we have traversed its entire circumference, that is, 2π radians. Thus 0 and 2π represent the same point on the unit circle. This is *not* to say that $0 = 2\pi$ (which is obviously untrue) but rather that traversing around the circle 2π radians brings us to the same point as traversing 0 radians. Similarly, traversing $\frac{\pi}{2}$ radians brings us to the same point as $\frac{\pi}{2} + 2\pi = \frac{5\pi}{2}$ radians, etc.

The right side Figure 3.2 shows multiples of 30°. Because 30° is one twelfth of 360°, the radian measure of a 30° angle is one twelfth the circumference 2π of the unit circle, that is, 30° is $\frac{1}{12}2\pi = \frac{\pi}{6}$ radians. The figure shows other multiples of 30°. Likewise, 60° (twice 30°) is $2\frac{\pi}{6} = \frac{\pi}{3}$, etc. Recall that we associate traversing counter-clockwise around the circle with *positive* radian measure. Traversing clockwise is interpreted as *negative* radian measures, as indicated in the figure. Thus (for instance) π and $-\pi$ bring us to the same point on the unit circle, as do $\frac{7\pi}{6}$ and $-\frac{5\pi}{6}$.

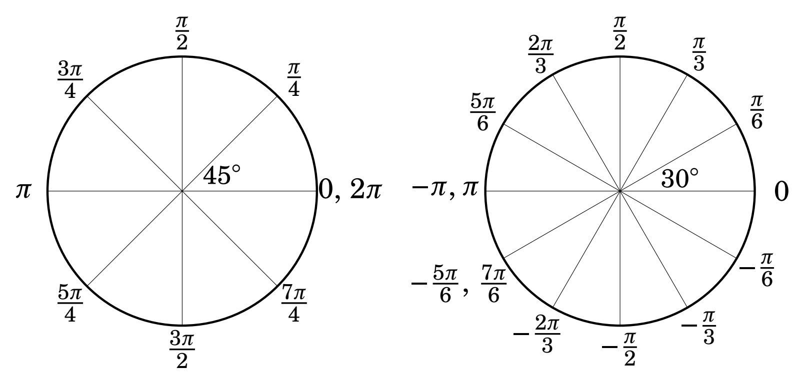


Figure 3.2. Some common angles (multiples of 45° and 30°) in radians.

It is of utmost importance to internalize (not just memorize) the diagrams in Figure 3.2. They provide a mental model that allows us to quickly convert between degrees and radians for angles that are integer multiples of 45 or 30 degrees. We will need to do this often. On occasion we may need to convert other angles, and again there is a simple mental model that can be used for this.

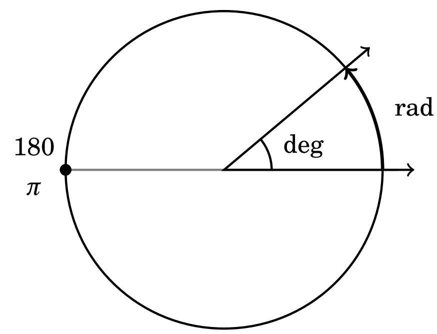
It is easy convert between radians and degrees by keeping the following picture in mind. The angle has degree measure "deg" and radian measure "rad." Since 180 degrees is π radians, the following ratios are equal:

$$\frac{\text{deg}}{180} = \frac{\text{rad}}{\pi}.$$

Solving two ways, we get

$$\text{deg} = \text{rad}\frac{180}{\pi},$$

$$\text{rad} = \text{deg}\frac{\pi}{180}.$$



Example 3.1 Convert 40° and 120° to radians, and $\pi/5$ radians to degrees.

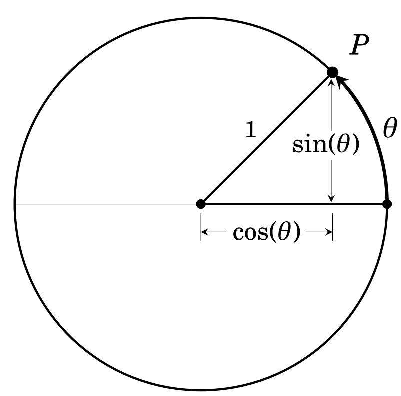
By the above formula, a 40° degree angle has radian measure $40\frac{\pi}{180} = \frac{2\pi}{9}$. Also 120° is $120\frac{\pi}{180} = \frac{2\pi}{3}$ radians. (This also follows very simply from the right side of Figure 3.2.) Finally, $\pi/5$ radians is $\frac{\pi}{5}\frac{180}{\pi} = 36$ degrees. ✍

Having reviewed radian measure, we now recall the definition of the two trigonometric functions sine and cosine, abbreviated as sin and cos. The values of these functions can be read straight off the unit circle.

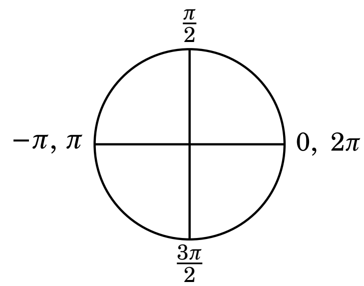
Definition 3.1 Given a real number θ, let P be the point at θ radians on the unit circle, as indicated on the right. The functions sin and cos are defined as

$$\begin{aligned} \cos(\theta) &= x\text{-coordinate of the point } P, \\ \sin(\theta) &= y\text{-coordinate of the point } P. \end{aligned}$$

As θ can be any real number, functions sin and cos both have domain $\mathbb{R}$.



This definition, coupled with our knowledge of the unit circle, makes it easy to mentally find sin or cos of any integer multiple of $\frac{\pi}{2}$. Just read the x- or y-coordinates off the unit circle. The diagram on the right reveals:



$$\cos(0)=1,\quad \cos(\tfrac{\pi}{2})=0,\quad \cos(\pi)=-1,$$
$$\sin(0)=0,\quad \sin(\tfrac{\pi}{2})=1,\quad \sin(\pi)=0.$$

Also $\cos(-\pi)=-1$, $\sin(\frac{3\pi}{2})=-1$, and $\cos(\frac{3\pi}{2})=0$. As $\frac{7\pi}{2}$ and $\frac{3\pi}{2}$ are at the same point on the unit circle, $\cos(\frac{7\pi}{2})=\cos(\frac{3\pi}{2})=0$. Avoid using a calculator for such simple computations. Working them out with the unit circle reinforces their meaning; a calculator invites us to forget the meaning.

To compute sin and cos of many other angles, it is helpful to know the two right triangles in Figure 3.3. The 45-45-90 triangle has a hypotenuse of length 1 and two legs of length $\frac{\sqrt{2}}{2}$. (Numbers that are easily gotten from the Pythagorean theorem.) The 30-60-90 triangle is half of an equilateral triangle with all sides of length 1. Thus one leg has length $\frac{1}{2}$, and the Pythagorean theorem yields $\frac{\sqrt{3}}{2}$ for the other.

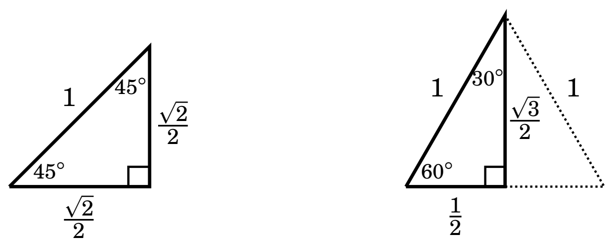
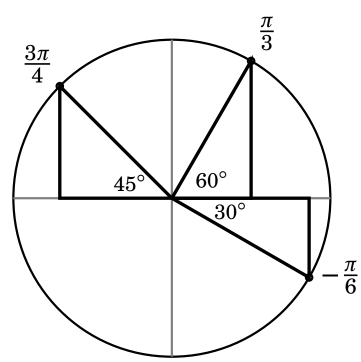


Figure 3.3. Standard triangles: the 45-45-90 (left), and 30-60-90 (right).

These triangles help us find sin and cos of many angles. For instance, let's find sin and cos of $\frac{\pi}{3}$. The point on the unit circle at $\frac{\pi}{3}$ is the corner of a 30-60-90 triangle; we read off



$$\cos\left(\tfrac{\pi}{3}\right)=\tfrac{1}{2},\quad \sin\left(\tfrac{\pi}{3}\right)=\tfrac{\sqrt{3}}{2}.$$

Similarly the 45-45-90 triangle at $\frac{3\pi}{4}$ yields

$$\cos\left(\tfrac{3\pi}{4}\right)=-\tfrac{\sqrt{2}}{2},\quad \sin\left(\tfrac{3\pi}{4}\right)=\tfrac{\sqrt{2}}{2}.$$

The picture also tells us $\cos\left(-\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$ and $\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$. In this way we can compute sin and cos of any angle that is an integer multiple of $\frac{\pi}{4}$ or $\frac{\pi}{6}$.

Of course, if we were ever confronted with something like, say, $\sin\left(\frac{\pi}{11}\right)$, we would be stuck because none of our standard triangles fit this problem, as $\frac{\pi}{11}$ is not a multiple of $\frac{\pi}{4}$ or $\frac{\pi}{6}$. As a last resort we could take out a calculator and get an approximate answer to ten or so decimal places. But it is a fact that multiples of $\frac{\pi}{4}$ and $\frac{\pi}{6}$ come up often, especially in college-level mathematics courses like this one. Understanding how to handle them gives us a conceptual framework for trigonometric functions that is necessary for further study and applications.

Some comments on notation are in order. It is common to abbreviate $\sin(\theta)$ and $\cos(\theta)$ as $\sin\theta$ and $\cos\theta$, even though this is like writing $f(x)$ as fx. At first glance it seems that this could introduce ambiguity. Does $\sin 2\theta + \pi$ mean $\sin(2\theta+\pi)$ or $\sin(2\theta)+\pi$, or $\sin(2)\theta+\pi$? The convention is that the trigonometric function applies to the term immediately following it. Thus $\sin 2\theta+\pi$ means $\sin(2\theta)+\pi$. If we intended it to mean $\sin(2\theta+\pi)$, then the parentheses would be mandatory. In the interest of clarity, this text will lean towards inserting parentheses whether or not they are necessary; however, there will certainly be many occasions where we omit them.

Also, expressions such as $(\sin(\theta))^2$ are abbreviated as $\sin^2\theta$ or $\sin^2(\theta)$. Thus, for instance, $\cos^4\frac{\pi}{4} = \left(\cos\frac{\pi}{4}\right)^4 = \left(\frac{\sqrt{2}}{2}\right)^4 = \frac{1}{4}$. The one exception to this rule concerns the exponent -1. $\cos^{-1}\theta$ does **not** mean $\frac{1}{\cos\theta}$, but rather the inverse cosine function $\arccos(x)$. (More on that in Chapter 6.)

Finally, we have been using θ for the argument of our trigonometric functions because it is associated with radian measure. Obviously, any variable would suffice. We do not balk at, say, $\cos x$, even though the argument x conflicts slightly with the definition of cos. In $\cos x$, the value of x is interpreted as a radian measure on the unit circle. The function value $\cos x$ is the x-coordinate of that point on the unit circle.

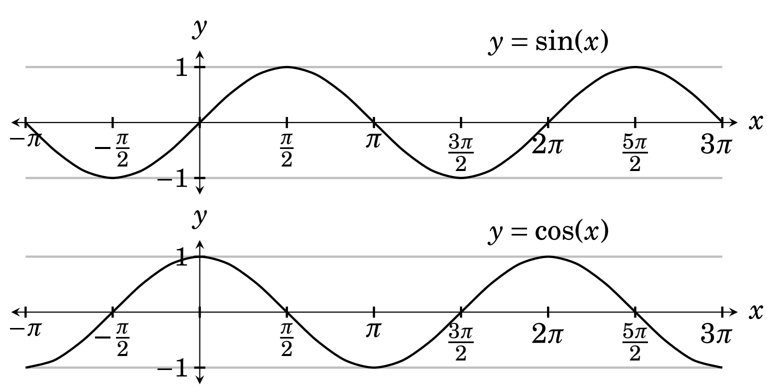


Figure 3.4. The graphs of $y = \sin(x)$ and $y = \cos(x)$

Of course the sine and cosine functions can be graphed, as in Figure 3.4. You are undoubtedly already familiar with these graphs, but take a moment to see how they follow from the definitions of $\sin(x)$ and $\cos(x)$ given here.

There are a total of six trigonometric functions, and each is expressed in terms of sine and cosine. Two of these are the secant and cosecant functions. They are abbreviated as $\sec(x)$ and $\csc(x)$, and defined as

$$\begin{aligned} \sec(x) &= \frac{1}{\cos(x)}, \\ \csc(x) &= \frac{1}{\sin(x)}. \end{aligned}$$

Figure 3.5 shows their graphs. (For reference, the graphs of $\cos(x)$ and $\sin(x)$ are shown dotted.)

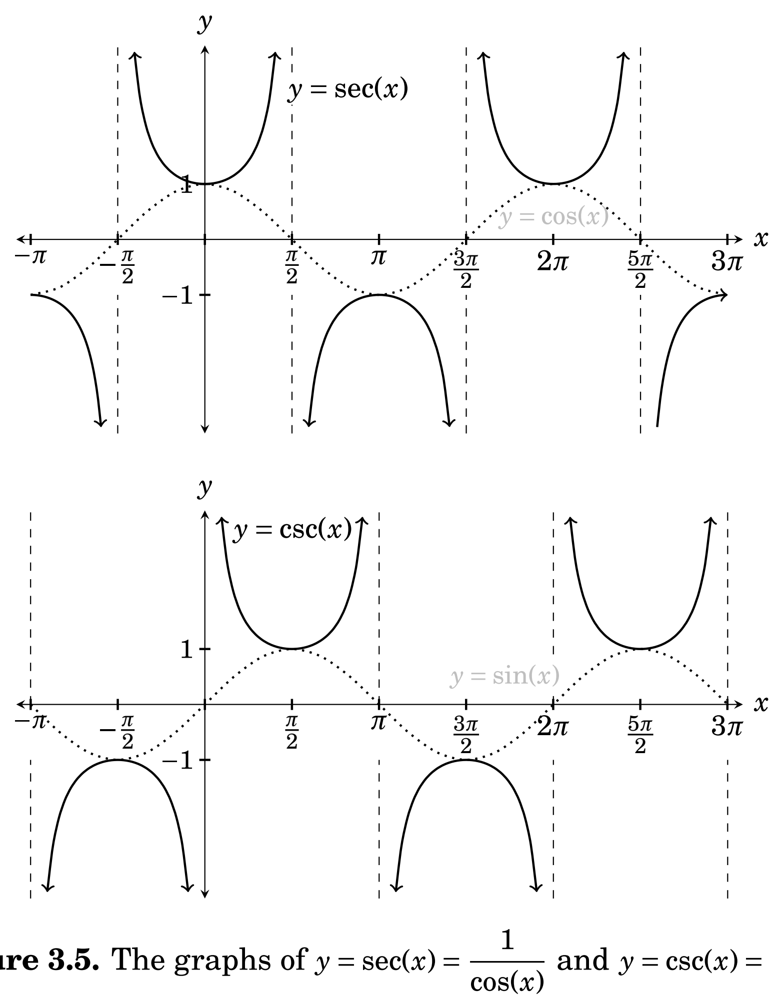


Figure 3.5. The graphs of $y = \sec(x) = \dfrac{1}{\cos(x)}$ and $y = \csc(x) = \dfrac{1}{\sin(x)}$

Take a moment to see how these graphs make sense. The value $y = \csc(x)$ is very large when its denominator $\sin(x)$ is close to 0, and undefined if $\sin(x) = 0$. Thus the graph has a vertical asymptote at $x = k\pi$, for any integer k. Similarly $y = \sec(x)$ has an asymptote wherever its denominator $\cos(x)$ is zero, namely at the points $\frac{\pi}{2} + k\pi$.

The final two trigonometric functions are called tangent and cotangent, and denoted as $\tan(x)$ and $\cot(x)$. They are defined as

$$\begin{aligned} \tan(x) &= \frac{\sin(x)}{\cos(x)}, \\ \cot(x) &= \frac{\cos(x)}{\sin(x)}. \end{aligned}$$

Figure 3.6 shows their graphs. As usual, you should examine the graphs to see that they conform to the above definitions.

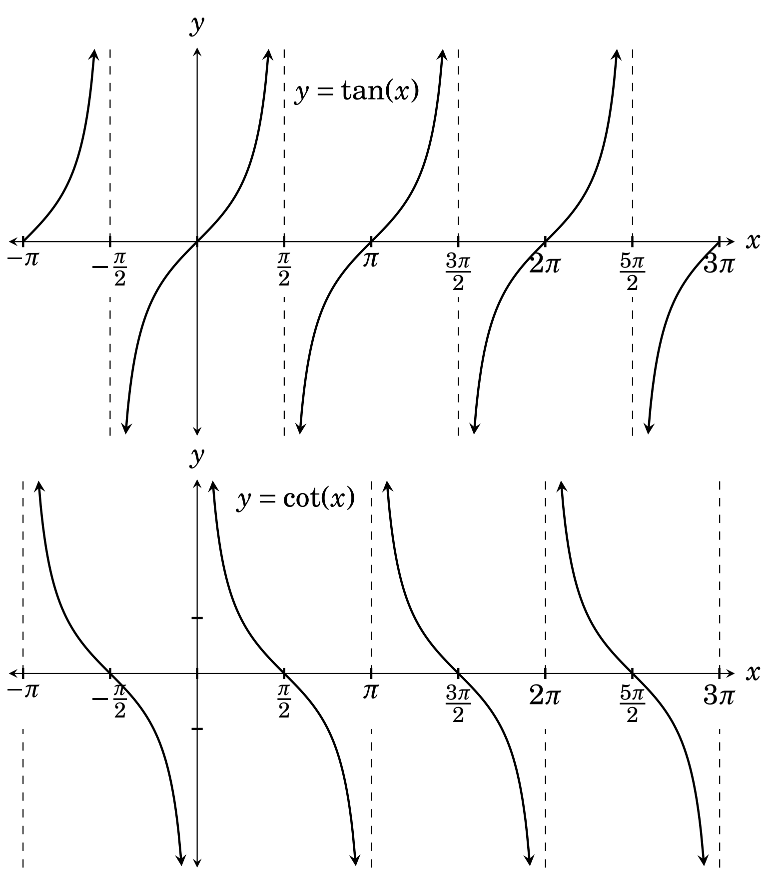


Figure 3.6. The functions $y = \tan(x)$ and $y = \cot(x)$.

If k is an integer, then the point at θ on the unit circle is identical to the point at $\theta + 2k\pi$. (Starting at θ, take k laps of length 2π around the unit circle to get to $\theta + k2\pi$.) Therefore

$$\cos(\theta + 2k\pi) = \cos(\theta), \tag{3.1}$$
$$\sin(\theta + 2k\pi) = \sin(\theta), \tag{3.2}$$
$$\sec(\theta + 2k\pi) = \sec(\theta), \tag{3.3}$$
$$\csc(\theta + 2k\pi) = \csc(\theta). \tag{3.4}$$

These equations are an algebraic manifestation of the fact that the graphs of cos, sin, sec and cos repeat themselves every 2π units. We express this in words by saying that these functions are *periodic*, with period 2π.

Equations similar to the above hold for tan and cot, but in those cases we can omit the factor of 2. A glance at their graphs (page 39) reveals that

$$\tan(\theta + k\pi) = \tan(\theta), \tag{3.5}$$
$$\cot(\theta + k\pi) = \cot(\theta). \tag{3.6}$$

Thus tan and cot are periodic with period π.

Test your understanding by working some of the following exercises. Do them without a calculator. With a calculator they are merely busy work, but doing them mentally reinforces our understanding of their *meaning*.

Exercises for Section 3.1

1. $\sin\frac{5\pi}{4}$ **2.** $\sin\frac{4\pi}{3}$ **3.** $\sin -\frac{5\pi}{3}$ **4.** $\sin -\frac{5\pi}{6}$ **5.** $\sin\frac{3\pi}{4}$ **6.** $\cos\frac{5\pi}{4}$

7. $\cos\frac{4\pi}{3}$ **8.** $\cos -\frac{5\pi}{3}$ **9.** $\cos -\frac{5\pi}{6}$ **10.** $\cos\frac{3\pi}{4}$ **11.** $\sec\frac{5\pi}{4}$ **12.** $\sec\frac{4\pi}{3}$

13. $\sec -\frac{5\pi}{3}$ **14.** $\csc -\frac{5\pi}{6}$ **15.** $\csc\frac{3\pi}{4}$ **16.** $\csc\frac{5\pi}{4}$ **17.** $\csc\frac{4\pi}{3}$ **18.** $\csc -\frac{5\pi}{3}$

19. $\csc -\frac{5\pi}{6}$ **20.** $\csc\frac{3\pi}{4}$ **21.** $\tan\frac{5\pi}{4}$ **22.** $\tan\frac{4\pi}{3}$ **23.** $\tan -\frac{5\pi}{3}$ **24.** $\tan -\frac{5\pi}{6}$

25. $\tan\frac{3\pi}{4}$ **26.** $\cot\frac{5\pi}{4}$ **27.** $\cot\frac{4\pi}{3}$ **28.** $\cot -\frac{5\pi}{3}$ **29.** $\cot -\frac{5\pi}{6}$ **30.** $\cot\frac{3\pi}{4}$

31. Convert 56° to radians.

32. Convert 190° to radians.

33. Convert 40° to radians.

34. Convert −50° to radians.

35. Convert 108° to radians.

36. Convert 72° to radians.

37. Convert $\frac{3\pi}{10}$ radians to degrees.

38. Convert 1 radian to degrees.

39. Convert $\frac{\pi}{18}$ radians to degrees.

40. Convert $\frac{5\pi}{9}$ radians to degrees.

41. Convert $-\frac{7\pi}{36}$ radians to degrees.

42. Convert $\frac{13\pi}{180}$ radians to degrees.

3.2 Solving Triangles

Sometimes, in working with a right triangle, we'll know the measurements of one of its angles and one of its sides, and will need to find the length of another of its sides. This problem can always be solved with a trig function. The process of finding one (or both) of the unknown sides is called *solving the triangle*. We review this now.

Consider the triangle on the left of Figure 3.7. If we knew only the value of θ and that the hypotenuse had length 1, then the other two sides have to be $\sin(\theta)$ and $\cos(\theta)$, as illustrated. That was easy because the hypotenuse being 1 means that the triangle fits neatly onto the unit circle.

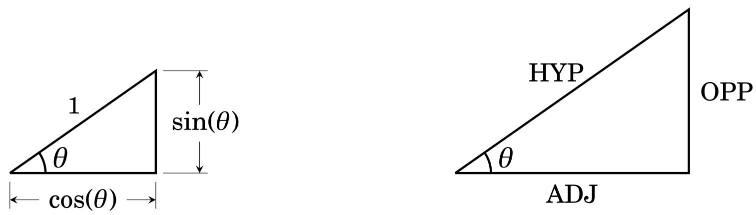


Figure 3.7. Similar triangles

More generally, the triangle could be as illustrated on the right. Here HYP stands for the length of the hypotenuse (not necessarily 1), OPP is the length of the side opposite θ, and ADJ is the side adjacent to θ. If $\text{HYP} \neq 1$, then OPP and ADJ are not equal to $\sin(\theta)$ and $\cos(\theta)$, as they are on the left. But certain ratios of them *are* equal to $\sin(\theta)$ and $\cos(\theta)$.

To see how, notice that the two triangles in Figure 3.7 are similar. Thus (for instance) we have $\frac{\text{OPP}}{\text{HYP}} = \frac{\sin(\theta)}{1} = \sin(\theta)$. Also, $\frac{\text{ADJ}}{\text{HYP}} = \frac{\cos(\theta)}{1} = \cos(\theta)$, and $\frac{\text{OPP}}{\text{ADJ}} = \frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta)$. In summary, for any right triangle we have

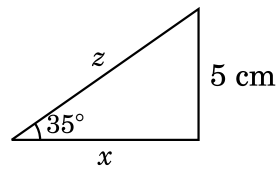
$$\sin(\theta) = \frac{\text{OPP}}{\text{HYP}}, \qquad \cos(\theta) = \frac{\text{ADJ}}{\text{HYP}}, \qquad \tan(\theta) = \frac{\text{OPP}}{\text{ADJ}}.$$

Reciprocating each of these yields formulas for the other trig functions:

$$\csc(\theta) = \frac{\text{HYP}}{\text{OPP}}, \qquad \sec(\theta) = \frac{\text{HYP}}{\text{ADJ}}, \qquad \cot(\theta) = \frac{\text{ADJ}}{\text{OPP}}.$$

This shows that each of the six trig functions is a certain ratio of the sides of a triangle with angle θ. If we know any two of the three quantities in one of these equations, then we can algebraically solve for the third. In this way we can solve for missing sides of right triangles.

Example 3.2 Find sides x and z of the triangle below. Also find its area.



To find z, notice that $\sin(35°) = \frac{\text{OPP}}{\text{HYP}} = \frac{5}{z}$. Reciprocating, $\csc(35°) = \frac{z}{5}$, so $z = 5\csc(35°)$. Now, if instead of 35° we had a nice angle like 30° or 45°, then we could work out the cosecant mentally. Here we are not so lucky, but we can get by with a calculator:

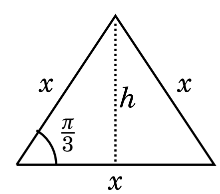
$$z = 5\csc(35°) \approx 5 \cdot 1.74344 = 8.71723\text{cm}.$$

To find x, notice that $\tan(35°) = \frac{\text{OPP}}{\text{ADJ}} = \frac{5}{x}$. Reciprocating, $\cot(35°) = \frac{x}{5}$, so $x = 5\cot(35°) \approx 7.14074$ cm.

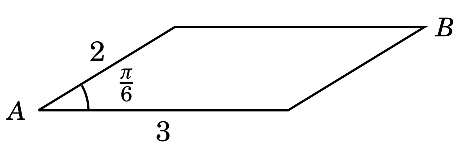
The area of the triangle is one half its base times its height, that is, $\text{Area} = \frac{1}{2}x \cdot 5 = \frac{1}{2}5\cot(35°) \cdot 5 = 25\cot(35°) \approx 35.7037001$ square cm. ✍

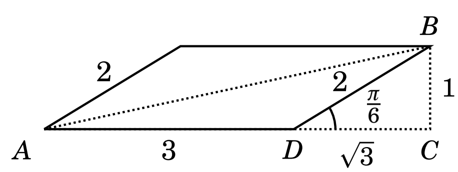
Example 3.3 Find the area of the equiangular triangle with sides of length x cm.

Solution: The answer will depend on x. The triangle is drawn on the right; each angle is 60°, or $\frac{\pi}{3}$ radians. Let h be the height of the triangle. To find h note that $\sin\frac{\pi}{3} = \frac{\text{OPP}}{\text{HYP}} = \frac{h}{x}$, so $h = x\sin\frac{\pi}{3} = x\frac{\sqrt{3}}{2}$ cm. Then the area of the triangle is $\frac{1}{2}bh = \frac{1}{2}xx\frac{\sqrt{3}}{2} = x^2\frac{\sqrt{3}}{4}$ square cm. ✍



Example 3.4 Find the length of the diagonal AB of the parallelogram shown below, left.

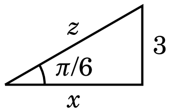




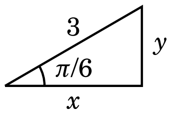
Solution: Drop a perpendicular from B to a point C, as shown on the right. Solving the triangle DBC, we get $BC = 2\sin\frac{\pi}{6} = 1$ and $DC = 2\cos\frac{\pi}{6} = \sqrt{3}$. Now, the diagonal AB is the hypotenuse of the right triangle ABC. By the Pythagorean theorem, we see that AB has length $\sqrt{(3+\sqrt{3})^2 + 1^2} = \sqrt{13 + 6\sqrt{3}}$ units. ✍

Exercises for Section 3.2

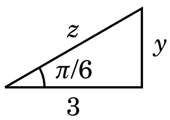
1. Find the missing sides.



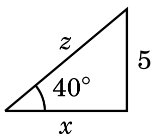
2. Find the missing sides.



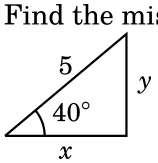
3. Find the missing sides.



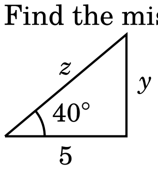
4. Find the missing sides.



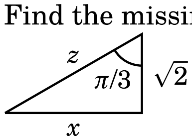
5. Find the missing sides.



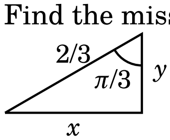
6. Find the missing sides.



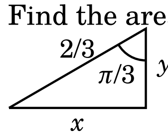
7. Find the missing sides.



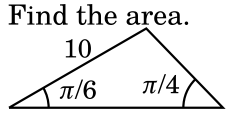
8. Find the missing sides.



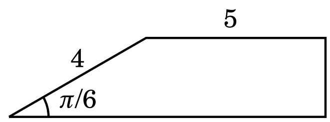
9. Find the area.



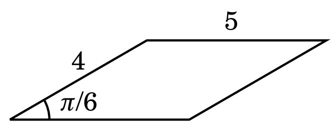
10. Find the area.



11. Find the area.

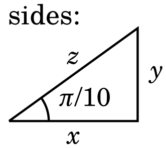


12. Find the area of the parallelogram.

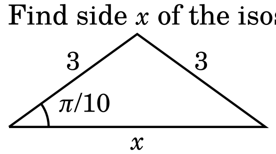


13. Find the lengths of both diagonals of the parallelogram in Exercise 12.

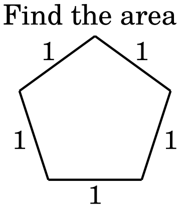
14. The area is 10 square units. Find the sides:



15. Find side x of the isosceles triangle:

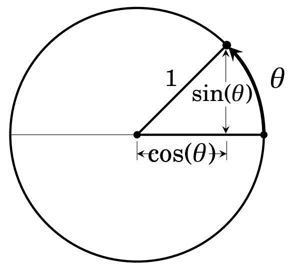


16. Find the area of the regular pentagon:



3.3 Trigonometric Identities

A **trigonometric identity** is an equation involving one or more trig functions that is true for all values of the variables that appear in it.



One important identity comes straight from the familiar diagram above. Applying the Pythagorean theorem to the right triangle yields

$$\sin^2(\theta) + \cos^2(\theta) = 1. \tag{3.7}$$

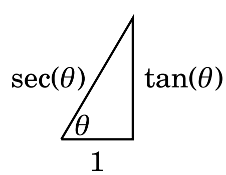
This equation — this *trigonometric identity* — holds for *any* value of θ. Dividing both sides of Equation (3.7) by $\cos^2(\theta)$ yields a *new* identity

$$\tan^2(\theta) + 1 = \sec^2(\theta). \tag{3.8}$$

Alternatively, we could divide both sides of Equation (3.7) by $\sin^2(\theta)$ to get

$$1 + \cot^2(\theta) = \csc^2(\theta). \tag{3.9}$$

Equation (3.7) is packed with meaning. It expresses a fundamental relationship between the sides of right triangles on the unit circle, and hence also a relationship between the functions sin and cos. By contrast, Equations (3.8) and (3.9) may seem less meaningful. But they too say something fundamental about right triangles. Consider the triangle below, whose adjacent side (not hypotenuse) has length 1.



For this triangle, OPP = OPP/1 = OPP/ADJ = $\tan(\theta)$, that is, its opposite side is $\tan(\theta)$. The hypotenuse is HYP = HYP/1 = HYP/ADJ = $\sec(\theta)$, as labeled. Equation (3.8) is just the Pythagorean theorem applied to this triangle. Similarly, Equation (3.9) can be understood as the Pythagorean theorem applied a triangle whose *opposite* side is 1, as you are invited to check.

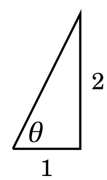
Example 3.5 Let θ be such that $\tan(\theta) = 2$ and $0 < \theta < \frac{\pi}{2}$. Find $\cos(\theta)$.

We give two solutions. The first uses Equation (3.8), which in the present situation says $2^2 + 1 = \sec^2(\theta)$, or

$$5 = \left(\frac{1}{\cos(\theta)}\right)^2.$$

Solving, $\cos(\theta) = \pm\frac{1}{\sqrt{5}}$. As $0 < \theta < \frac{\pi}{2}$, we have $\cos(\theta) > 0$. Thus $\cos(\theta) = \frac{1}{\sqrt{5}}$.

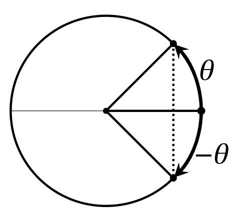
The second solution is more organic. Begin by drawing a triangle for which $\tan(\theta) = \frac{\text{OPP}}{\text{ADJ}} = \frac{2}{1} = 2$, like this one.



By the Pythagorean theorem, the hypotenuse has length $\text{HYP} = \sqrt{1^2 + 2^2} = \sqrt{5}$. Then $\cos(\theta) = \frac{\text{ADJ}}{\text{HYP}} = \frac{1}{\sqrt{5}}$, which agrees with our first solution.

This second solution has the advantage of quickly giving the values of all the other trigonometric functions at θ. For example $\sin(\theta) = \frac{\text{OPP}}{\text{HYP}} = \frac{2}{\sqrt{5}}$. It is good practice to use triangles (rather than blind reliance of formulas) when dealing with trigonometry. ✍

For two further identities, consider the two points located at θ and $-\theta$ on the unit circle, as illustrated below.



These points have the same x-coordinates, which is to say $\cos(-\theta) = \cos(\theta)$. But the y-coordinate of one is the negative of the y-coordinate of the other, which means $\sin(-\theta) = -\sin(\theta)$. In summary,

$$\cos(-\theta) = \cos(\theta), \tag{3.10}$$

$$\sin(-\theta) = -\sin(\theta). \tag{3.11}$$

At this point the unit circle should be so ingrained that the above equations are transparent. Indeed we will often use them without comment. But be careful not to read more into them than is actually there. For instance, if k is a number other than -1, it is generally **not true** that $\sin(k\theta) = k\sin(\theta)$.

You have probably seen the following **addition formulas** before. (Exercise 7 at the end of this section helps explain why they are true.)

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta) \tag{3.12}$$
$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta) \tag{3.13}$$

Putting $\alpha=\theta$ and $\beta=\theta$ in the Formulas (3.12) and (3.13) yields two **double angle formulas**, which will be useful several times in later chapters.

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta), \tag{3.14}$$
$$\cos(2\theta) = \cos^2(\theta)-\sin^2(\theta). \tag{3.15}$$

Identities can often be combined to good effect. For example Equations (3.10), (3.11) and (3.12) combine as

$$\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta).$$

Two other formulas come from adding (or subtracting) $1=\sin^2(\theta)+\cos^2(\theta)$ to (or from) Equation (3.15).

$$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}, \tag{3.16}$$
$$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}. \tag{3.17}$$

These can be useful because they allow us to replace a squared term with an expression with no square, thereby offering possibilities for simplification. They are sometimes called **half angle formulas** because they can be written as $\cos^2(\frac{\theta}{2})=\frac{1}{2}(1+\cos(\theta))$ and $\sin^2(\frac{\theta}{2})=\frac{1}{2}(1-\cos(\theta))$.

Example 3.6 Find $\sin\left(\frac{7\pi}{12}\right)$.

Solution: The value $\frac{7\pi}{12}$ is not one that lends an immediate evaluation. But note that $\frac{7\pi}{12}=\frac{\pi}{4}+\frac{\pi}{3}$. Using this and Equation (3.12),

$$\sin\left(\frac{7\pi}{12}\right) = \sin\left(\frac{\pi}{4}+\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right)+\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{2}}{2}\cdot\frac{1}{2}+\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{2}+\sqrt{6}}{4}.$$

Thus $\sin\left(\frac{7\pi}{12}\right)=\frac{\sqrt{2}+\sqrt{6}}{4}$. ✍

Here is a summary of our identities, including a few from the exercises.

Triangle Formulas

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ 1 + \tan^2\theta &= \sec^2\theta \\ 1 + \cot^2\theta &= \csc^2\theta \end{aligned}$$

Addition Formulas

$$\begin{aligned} \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \cos(\alpha+\beta) &= \cos^2\alpha - \sin^2\beta \\ \tan(\alpha+\beta) &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \end{aligned}$$

Double Angle Formulas

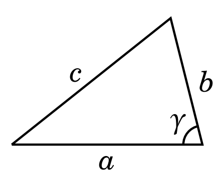
$$\begin{aligned} \sin 2\theta &= 2\sin\theta\cos\theta \\ \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ \tan 2\theta &= \frac{2\tan\theta}{1 - \tan^2\theta} \end{aligned}$$

Half Angle Formulas

$$\begin{aligned} \cos^2(\theta) &= \frac{1 + \cos(2\theta)}{2} \\ \sin^2(\theta) &= \frac{1 - \cos(2\theta)}{2} \end{aligned}$$

We close with a final trigonometric identity, the so-called *law of cosines*. It is something that may not be needed until the third semester of calculus. You may opt to skip it now and revisit it when (and if) the need arises.

The law of cosines is a generalization of the Pythagorean theorem in that it applies to triangles that are not necessarily right-angled. Consider the triangle below with three side lengths a, b and c. It would be a right triangle if $\gamma = \frac{\pi}{2}$, and in that case the Pythagorean theorem says $a^2 + b^2 = c^2$. The law of cosines relates a, b and c for any value of γ.



The Law of Cosines: If a triangle has sides of length a, b and c, and γ is the measure of the angle opposite c (as illustrated above), then

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma). \tag{3.18}$$

This reduces to the Pythagorean theorem when γ is a right angle, as then $\cos(\gamma) = 0$. The law of cosines is not hard to verify. See Exercise 9.

Exercises for Section 3.3

1. Use this chapter's addition formulas (3.12) and (3.13) to verify the two identities $\cos(\pi/2-\theta)=\sin(\theta)$ and $\sin(\pi/2-\theta)=\cos(\theta)$.

2. Use the unit circle (and your knowledge of rudimentary geometry) to explain why $\cos(\pi/2-\theta)=\sin(\theta)$ and $\sin(\pi/2-\theta)=\cos(\theta)$.

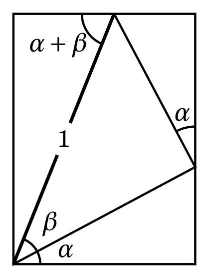
3. Verify the identity $\sin(\alpha)+\sin(\beta)=2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$.

4. Verify the identity $\cos(\alpha)+\cos(\beta)=2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$.

5. Verify the identity $\sin(\alpha)\sin(\beta)=\frac{1}{2}\big[\cos(\alpha-\beta)-\cos(\alpha+\beta)\big]$.

6. Verify the identity $\cos(\alpha)\cos(\beta)=\frac{1}{2}\big[\cos(\alpha-\beta)+\cos(\alpha+\beta)\big]$.

7. This exercise justifies the formulas $\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$ and $\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$. An outline follows.

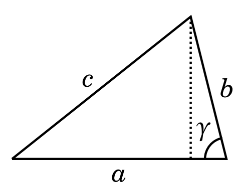


The rectangle on the right is divided into four right triangles; the heavy diagonal has length 1. Solve each triangle to find the lengths of its sides.

The fact that the right and left sides of the rectangle are equal lengths will give the first formula; the fact that the top and bottom sides are equal lengths gives the second.

8. Derive the addition formula $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$. (Suggestion: Start with Equations (3.12) and (3.12).)

9. Prove Equation (3.18), the law of cosines.



Suggestion: The relevant triangle is sketched to the right. Drop a perpendicular from the apex to the bottom side, as illustrated. This divides the triangle into two right triangles. Solve for their edges and apply the Pythagorean theorem to the triangle on the left.

The angle γ in the diagram is acute. Use a different diagram for obtuse angles.

10. There is a trigonometric identity called the **law of sines**, which states the following: Suppose a triangle has three angles α, β and γ. Further, suppose the side opposite α has length a, the side opposite β has length b, and the side opposite γ has length c. Then

$$\frac{\sin(\alpha)}{a}=\frac{\sin(\beta)}{b}=\frac{\sin(\gamma)}{c}.$$

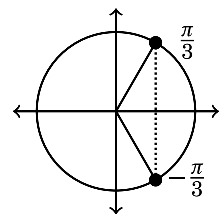
Prove this. (Consider drawing a diagram as in Exercise 9, above.)

3.4 Solving Trigonometric Equations

We just discussed *trigonometric identities*, equations with trig functions that are true for every possible value of the variable in the equation.

It is of course possible to have an equation that is true for *some* values of the variable but not true for others. Such an equation is not an identity. But it *is* something that we could *solve*, that is, find all values of the variable that make it true. We illustrate this with a series of examples.

Example 3.7 Solve the equation $\cos(\theta) = \dfrac{1}{2}$.



Solution: We seek all values of θ for which $\cos(\theta) = \frac{1}{2}$. For this, the unit circle is helpful. Our experience with it tells us that it has two points with x-coordinate $\frac{1}{2}$ (i.e., for which $\cos(\theta) = \frac{1}{2}$), and they are $\theta = \pm\frac{\pi}{3}$. Thus two solutions of the equation are $\theta \pm \frac{\pi}{3}$. But we have not yet found *all* solutions.

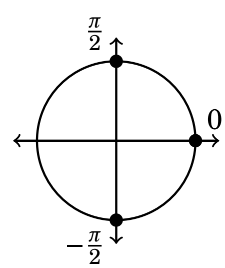
We must not forget that we could add any integer multiple of 2π to either of these θ values and still be at the same point on the unit circle. Thus the set of all solutions is $\theta = \pm\frac{\pi}{3} + 2\pi k$ where k is an integer.

Answer: The set of solutions is $\left\{\pm\dfrac{\pi}{3} + 2\pi k \,:\, k = 0, \pm 1, \pm 2, \pm 3, \ldots\right\}$. Notice that there are infinitely many solutions. This is typical with trigonometric equations. ✍

Example 3.8 Solve the equation $\cos^2(x) = \cos(x)$.

Solution: We seek all values of x that make this a true equation. Notice that for a first step we *cannot* divide both sides by $\cos(x)$. The reason is that $\cos(x)$ can be zero, in which case the division is illegal. (In fact, if our first step were to divide by $\cos(x)$, we would miss many of the solutions. Try it.) Instead we can avoid division by getting all terms to one side and factoring:

$$\begin{aligned} \cos^2(x) &= \cos(x) \\ \cos^2(x) - \cos(x) &= 0 \\ \cos(x)\big(\cos(x) - 1\big) &= 0 \end{aligned}$$



This holds when $\cos(x) = 0$ or $\cos(x) - 1 = 0$, that is, when $\cos(x)=0$ or $\cos(x)=1$. We analyze these cases separately.

There are two points on the unit circle at which $\cos(x) = 0$, namely at radians $x = \pm\frac{\pi}{2}$. Thus the factor $\cos(x)$ is zero exactly when $x = \pm\frac{\pi}{2} + 2\pi k$, for any integer k. A glance at the unit circle tells us we may write this in the slightly more compact form $x = \frac{\pi}{2} + \pi k$.

Now let's turn to $\cos(x) = 1$ (which makes the second factor 0). There is only one point on the unit circle at which $\cos(x) = 1$, namely $x = 0$. (See the diagram.) Thus $\cos(x) = 1$ provided that $x = 0 + 2\pi k$ for integers k.

The previous two paragraphs give our final answer. The set of all solutions to the equation is $\left\{\frac{\pi}{2} + \pi k \,:\, k = 0, \pm1, \pm2, \pm3, \ldots\right\}$ together with the set $\{2\pi k \,:\, k = 0, \pm1, \pm2, \pm3, \ldots\}$. ✍

Example 3.9 Find the domain of the function $f(x) = \dfrac{1}{1+\sin(x)}$.

Solution: This problem is not asking us to solve an equation, at least not directly. Looking at the function, we see that it is meaningful for any input x *except* those that make the denominator zero, that is, those x for which $1+\sin(x) = 0$. To find all such "bad" values of x, we need to solve the equation $1+\sin(x) = 0$, that is, we need to solve $\sin(x) = -1$. There is only one point on the unit circle for which $\sin(x) = -1$, and it is located at $\frac{3\pi}{2}$. It follows that the solutions of $\sin(x) = -1$ are the values $x = \frac{3\pi}{2} + 2\pi k$, where k is an integer. These are the values of x that are not in the domain of $f(x)$.

Answer: The domain of $f(x)$ is the set of all numbers *except* those of form $x = \frac{3\pi}{2} + 2\pi k$, that is, $\left\{x \in \mathbb{R} \,:\, x \neq \dfrac{3\pi}{2} + 2\pi k, \text{ where } k = 0, \pm1, \pm2, \pm3, \ldots\right\}$.

Note: There are many slightly different (but equally correct) ways of formulating this answer. For example, we could said that the point on the unit circle for which $\sin(x) = -1$ is located at $x = -\frac{\pi}{2}$. Then for the domain we would get all numbers *except* those of form $-\frac{\pi}{2} + 2\pi k$. Although it looks different, this is the same answer as above. Note that $-\frac{\pi}{2} = \frac{3\pi}{2} - 2\pi$, so adding any integer multiple of 2π to $-\frac{\pi}{2}$ gives $\frac{3\pi}{2}$ plus some other integer multiple of 2π, and conversely. ✍

In solving a trigonometric equation, we often arrive at one or more simpler equations in a form such as $\cos(\theta) = \frac{1}{2}$. We then have to think backwards and ask ourselves "For what θ is $\cos(\theta) = \frac{1}{2}$?" The unit circle usually provides an answer.

But it is of course possible that we might arrive at an equation such as $\cos(\theta) = \frac{1}{3}$. Here we are stuck because no familiar angle θ satisfies this equation. Chapter 6 gives a solution to this dilemma. It introduces the *inverse trigonometric functions*. One of these functions is $\cos^{-1}(x)$, which equals an angle θ for which $\cos(\theta) = x$. Thus a solution to $\cos(\theta) = \frac{1}{3}$ is $\theta = \cos^{-1}(1/3)$.

We will cover this in good time. No such inverse trig functions are needed in the following exercises. The unit circle is enough.

Exercises for Section 3.4

Solve the equations.

1. $1 + \tan(\theta) = 0$

2. $1 - \tan^2(\theta) = 0$

3. $4\sin^2(\theta) = 3$

4. $2\sin^2(\theta) - \sin(\theta) = 1$

5. $2\cos^2(\theta) - 1 = 0$

6. $\sin^2(x) = \sin(x)$.

7. $2\sin(\theta)\cos(\theta) = -1$

8. $\sin(\theta)\cos(\theta) = \sin(2\theta)$.
Hint: Use Equation (3.14).

9. $\cos^2(\theta) - \sin^2(\theta) = \sin(2\theta)$

10. $\sin^2(x) = -\cos(2x)$ (Use Equation (3.17).)

11. $\sin^2(x) - 1 = \cos(x)$

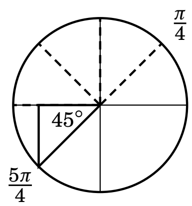
12. $\sin(x) = \sqrt{3}\cos(x)$

13. $\tan^2(x) = 3$

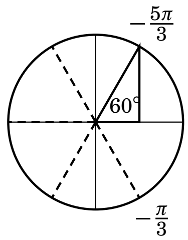
3.5 Exercise Solutions for Chapter 3

Solutions for Section 3.1

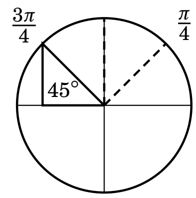
1. $\sin \frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$



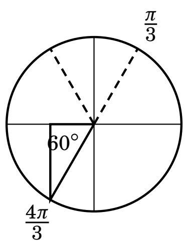
3. $\sin -\frac{5\pi}{3} = \frac{\sqrt{3}}{2}$



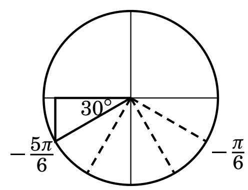
5. $\sin \frac{3\pi}{4} = \frac{\sqrt{2}}{2}$



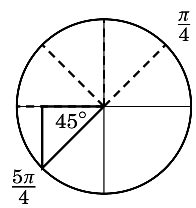
7. $\cos \frac{4\pi}{3} = -\frac{1}{2}$



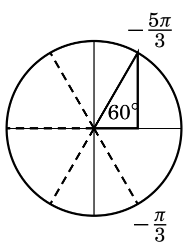
9. $\cos -\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$



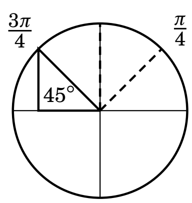
11. $\sec \frac{5\pi}{4} = \frac{1}{\cos \frac{5\pi}{4}} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$



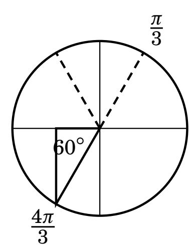
13. $\sec -\frac{5\pi}{3} \frac{1}{\cos -\frac{5\pi}{3}} = \frac{1}{1/2} = 2$



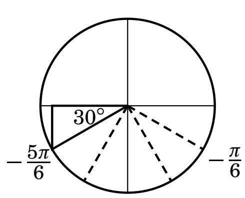
15. $\csc \frac{3\pi}{4} = \frac{1}{\sin \frac{3\pi}{4}} = \frac{2}{\sqrt{2}} = \sqrt{2}$



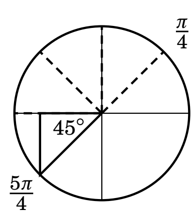
17. $\csc \frac{4\pi}{3} = \frac{1}{\sin \frac{4\pi}{3}} = -\frac{2}{\sqrt{3}}$



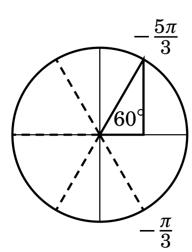
19. $\csc -\frac{5\pi}{6} = \frac{1}{\sin -\frac{5\pi}{6}} = -2$



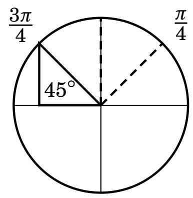
21. $\tan \frac{5\pi}{4} = \frac{\sin \frac{5\pi}{4}}{\cos \frac{5\pi}{4}} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2} = 1$



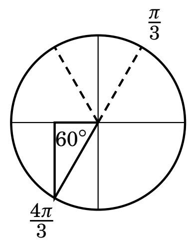
23. $\tan -\frac{5\pi}{3} = \frac{\sin \frac{5\pi}{3}}{\cos \frac{5\pi}{3}} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$



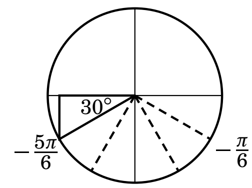
25. $\tan\frac{3\pi}{4} = \frac{\sin\frac{3\pi}{4}}{\cos\frac{3\pi}{4}} = \frac{\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = -1$



27. $\cot\frac{4\pi}{3} = \frac{\cos\frac{4\pi}{3}}{\sin\frac{4\pi}{3}} = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \sqrt{3}$



29. $\cot -\frac{5\pi}{6} = \frac{\cos -\frac{5\pi}{6}}{\sin -\frac{5\pi}{6}} = \frac{\frac{-\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3}$



31. Convert 56° to radians.
rad $= \frac{56}{180}\pi = \frac{14}{45}\pi$

33. Convert 40° to radians.
rad $= \frac{40}{180}\pi = \frac{2}{9}\pi$

35. Convert 108° to radians.
rad $= \frac{108}{180}\pi = \frac{27}{45}\pi$

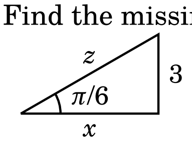
37. Convert $\frac{3\pi}{10}$ radians to degrees.
deg $= \frac{3\pi/10}{\pi}180 = 54°$

39. Convert $\frac{\pi}{18}$ radians to degrees.
deg $= \frac{\pi/18}{\pi}180 = 10°$

41. Convert $-\frac{7\pi}{36}$ radians to degrees.
deg $= \frac{-7\pi/36}{\pi}180 = -35°$

Exercises for Section 3.2

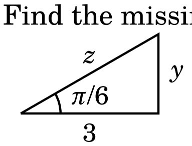
1. Find the missing sides.



To find z: $\sin\frac{\pi}{6} = \frac{\text{OPP}}{\text{HYP}}$, so $\frac{1}{2} = \frac{3}{z}$. Therefore $\boxed{z = 6.}$

To find x: $\tan\frac{\pi}{6} = \frac{\text{OPP}}{\text{ADJ}}$, so $\frac{1}{\sqrt{3}} = \frac{3}{x}$. Therefore $\boxed{x = 3\sqrt{3}.}$

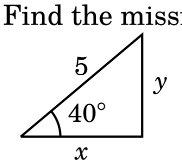
3. Find the missing sides.



To find z: $\cos\frac{\pi}{6} = \frac{\text{ADJ}}{\text{HYP}}$, so $\frac{\sqrt{3}}{2} = \frac{3}{z}$. Therefore $\boxed{z = \frac{6}{\sqrt{3}}.}$

To find y: $\tan\frac{\pi}{6} = \frac{\text{OPP}}{\text{ADJ}}$, so $\frac{1}{\sqrt{3}} = \frac{y}{3}$. Therefore $\boxed{y = \frac{3}{\sqrt{3}}.}$

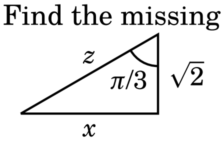
5. Find the missing sides.



To find x: $\cos 40° = \frac{\text{ADH}}{\text{HYP}}$, so $\cos 40° = \frac{x}{5}$, so $\boxed{x = 5\cos 40°.}$

To find y: $\sin 40° = \frac{\text{OPP}}{\text{HYP}}$, so $\sin 40° = \frac{y}{5}$, so $\boxed{y = 5\sin 40°.}$

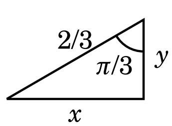
7. Find the missing sides.



To find x: $\tan\frac{\pi}{3} = \frac{\text{OPP}}{\text{ADJ}}$, so $\sqrt{3} = \frac{x}{\sqrt{2}}$. Thus $\boxed{x = \sqrt{6}.}$

To find z: $\cos\frac{\pi}{6} = \frac{\text{ADJ}}{\text{HYP}}$, so $\frac{1}{2} = \frac{\sqrt{2}}{z}$. Thus $\boxed{z = 2\sqrt{2}.}$

9. Find the area.

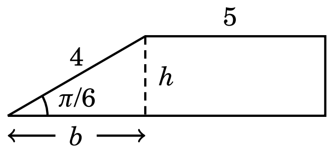


To find x: $\sin\frac{\pi}{3} = \frac{\text{OPP}}{\text{HYP}}$. Thus $\frac{\sqrt{3}}{2} = \frac{x}{2/3}$, so $\boxed{x = \frac{\sqrt{3}}{3}.}$

To find y: $\cos\frac{\pi}{3} = \frac{\text{ADJ}}{\text{HYP}}$. Thus $\frac{1}{2} = \frac{y}{2/3}$, so $\boxed{y = \frac{1}{3}.}$

Area $= \frac{1}{2}\text{base}\cdot\text{height} = \frac{1}{2}xy = \frac{1}{2}\cdot\frac{\sqrt{3}}{3}\cdot\frac{1}{3} = \boxed{\frac{\sqrt{3}}{18}\text{ square units.}}$

11. Find the area.



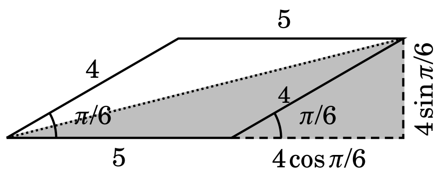
To find height h: $\frac{1}{2} = \sin\frac{\pi}{6} = \frac{\text{OPP}}{\text{HYP}} = \frac{h}{4}$, so $\boxed{h = 2.}$

For base b: $\frac{\sqrt{3}}{2} = \cos\frac{\pi}{6} = \frac{\text{ADJ}}{\text{HYP}} = \frac{b}{4}$, so $\boxed{b = 2\sqrt{3}.}$

Area = (triangle area) + (rectangle area)

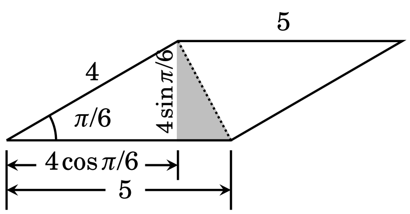
$= \frac{1}{2}2\cdot 2\sqrt{3} + 5\cdot 2 = \boxed{2\sqrt{3} + 10 \text{ square units.}}$

13. Find the lengths of both diagonals of the parallelogram in Exercise 12.



The longer diagonal is the hypotenuse of the shaded right triangle above, which by the Pythagorean theorem is $\sqrt{(5+4\cos(\pi/6))^2 + 4\sin^2(\pi/6)} = \sqrt{\left(5+4\cdot\frac{\sqrt{3}}{2}\right)^2 + \left(4\cdot\frac{1}{2}\right)^2}$

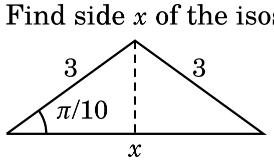
$= \sqrt{(5+2\sqrt{3})^2 + 2^2} = \sqrt{25+10\sqrt{3}+12+4} = \boxed{\sqrt{41+10\sqrt{3}}.}$

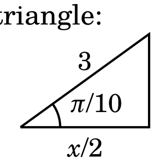


The shorter diagonal is the hypotenuse of the shaded right triangle above, which by the Pythagorean theorem is $\sqrt{(5-4\cos(\pi/6))^2 + 4\sin^2(\pi/6)} = \sqrt{\left(5-4\cdot\frac{\sqrt{3}}{2}\right)^2 + \left(4\cdot\frac{1}{2}\right)^2}$

$= \sqrt{(5-2\sqrt{3})^2 + 2^2} = \sqrt{25-10\sqrt{3}+12+4} = \boxed{\sqrt{41-10\sqrt{3}}.}$

15. Find side x of the isosceles triangle:





Cut the triangle in half by dropping a perpendicular bisector to its base as indicated. Then looking at half the triangle (above, right), we get

$\cos(\pi/10) = \frac{\text{ADJ}}{\text{HYP}} = \frac{x/2}{3} = \frac{x}{6}$. Thus $x = 6\cos(\pi/10) \approx 5.70633909777$ (with calculator).

Solutions for Section 3.3

1. Use the addition formulas (3.12) and (3.13) to establish the two identities $\cos(\pi/2-\theta)=\sin(\theta)$ and $\sin(\pi/2-\theta)=\cos(\theta)$.

First, $\cos(\pi/2-\theta)=\cos\big(\pi/2+(-\theta)\big)=\cos(\pi/2)\cos(-\theta)-\sin(\pi/2)\sin(-\theta)$
$=0\cdot\cos(-\theta)-1\cdot\sin(-\theta)=-\sin(-\theta)=\sin(\theta).$

Next, $\sin(\pi/2-\theta)=\sin\big(\pi/2+(-\theta)\big)=\sin(\pi/2)\cos(-\theta)-\cos(\pi/2)\sin(-\theta)$
$=1\cdot\cos(-\theta)-0\cdot\sin(-\theta)=\cos(-\theta)=\cos(\theta).$

3. Verify the identity $\sin(\alpha)+\sin(\beta)=2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$.

Using the addition formulas (3.12) and (3.13), the right-hand side becomes

$$
\begin{aligned}
& 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\\
= \;& 2\sin\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)\cos\left(\frac{\alpha}{2}+\frac{-\beta}{2}\right)\\
= \;& 2\left(\sin\frac{\alpha}{2}\cos\frac{\beta}{2}+\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\right)\left(\cos\frac{\alpha}{2}\cos\frac{-\beta}{2}-\sin\frac{\alpha}{2}\sin\frac{-\beta}{2}\right)\\
= \;& 2\left(\sin\frac{\alpha}{2}\cos\frac{\beta}{2}+\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\right)\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2}+\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right)\\
= \;& 2\left(\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\cos^2\frac{\beta}{2}+\sin^2\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\beta}{2}+\cos^2\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\beta}{2}+\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\sin^2\frac{\beta}{2}\right)\\
= \;& 2\left(\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\left(\cos^2\frac{\beta}{2}+\sin^2\frac{\beta}{2}\right)\;+\;\sin\frac{\beta}{2}\cos\frac{\beta}{2}\left(\cos^2\frac{\beta}{2}+\sin^2\frac{\beta}{2}\right)\right)\\
= \;& 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\;+\;\sin\frac{\beta}{2}\cos\frac{\beta}{2}\\
= \;& \sin 2\cdot\frac{\alpha}{2}+\sin 2\cdot\frac{\beta}{2}\;=\;\sin(\alpha)+\sin(\beta).
\end{aligned}
$$

(In the last line we applied the double angle formula (3.14).)

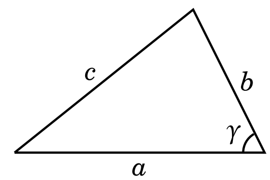
5. Verify the identity $\sin(\alpha)\sin(\beta)=\frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$.

Using the addition formulas (3.12) and (3.13), the right-hand side becomes

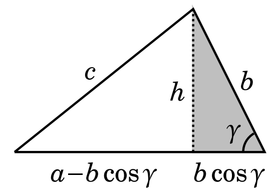
$$
\begin{aligned}
& \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]\\
= \;& \frac{1}{2}\left[\cos(\alpha+(-\beta))-\cos(\alpha+\beta)\right]\\
= \;& \frac{1}{2}\Big[\big(\cos(\alpha)\cos(-\beta)-\sin(\alpha)\sin(-\beta)\big)-\big(\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)\big)\Big]\\
= \;& \frac{1}{2}\Big[\big(\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)\big)-\big(\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)\big)\Big]\\
= \;& \sin(\alpha)\sin(\beta).
\end{aligned}
$$

9. Prove Equation (3.18), the law of cosines.

First assume γ is acute, so our triangle is as pictured on the right. We need to show that $c^2 = a^2 + b^2 - 2ab\cos(\gamma)$



Drop a perpendicular from the apex, cutting the triangle into two smaller triangles, shown white and gray on the right. By standard trigonometry, the gray triangle has base $b\cos\gamma$. Then the white triangle has base $a - b\cos\gamma$. Call the height of both triangles h.

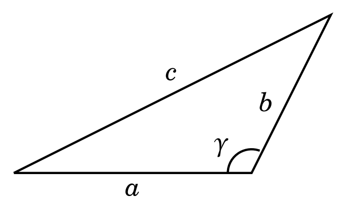


By the Pythagorean theorem applied to the white triangle, $h^2 = c^2 - (a - b\cos\gamma)^2$. By the Pythagorean theorem applied to the gray triangle, $h^2 = b^2 - (b\cos\gamma)^2$. From these two equations we get

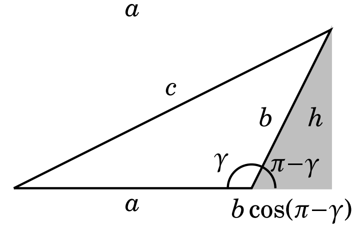
$$\begin{aligned} c^2 - (a - b\cos\gamma)^2 &= b^2 - (b\cos\gamma)^2 \\ c^2 - a^2 + 2ab\cos\gamma - b^2\cos^2\gamma &= b^2 - b^2\cos^2\gamma \\ c^2 &= a^2 + b^2 - 2ab\cos\gamma, \end{aligned}$$

which is what we needed to show.

Next assume γ is obtuse, so our triangle is as shown on the right. Again we need to show that $c^2 = a^2 + b^2 - 2ab\cos(\gamma)$



Drop a perpendicular from the apex, forming a new triangle on the right, shown gray. One angle of the gray triangle is $\pi - \gamma$, as indicated. By standard trigonometry, this gray triangle has base $b\cos(\pi - \gamma)$. Call its height h.



By the Pythagorean theorem applied to the gray triangle, $h^2 = b^2 - (b\cos(\pi-\gamma))^2$. There is also a large right triangle that is the union of the gray triangle with the original (white) triangle. the Pythagorean theorem applied to *it* yields $h^2 = c^2 - (a + b\cos(\pi-\gamma))^2$. Together these two equations give

$$\begin{aligned} c^2 - (a + b\cos(\pi-\gamma))^2 &= b^2 - (b\cos(\pi-\gamma))^2 \\ c^2 - a^2 - 2ab\cos(\pi-\gamma) - b^2\cos^2(\pi-\gamma) &= b^2 - b^2\cos^2(\pi-\gamma) \\ c^2 &= a^2 + b^2 + 2ab\cos(\pi-\gamma), \end{aligned}$$

Noting that $\cos(\pi-\gamma) = -\cos(\gamma)$, this becomes $c^2 = a^2 + b^2 - 2ab\cos(\gamma)$.

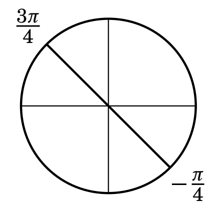
Exercises for Section 3.4

1. Solve $1+\tan(\theta)=0$.

Write this as $\tan(\theta)=-1$.

Note that $\tan(-\pi/4)=-1$, and adding any integer multiple of π to $-\pi/4$ brings us to a point $k\pi-\frac{\pi}{4}$ for which $\tan\left(k\pi-\frac{\pi}{4}\right)=-1$. Therefore the solutions of this equation are all θ for which

$\boxed{\theta=k\pi-\frac{\pi}{4},\text{ where } k=0,\pm1,\pm2,\pm3,\ldots.}$



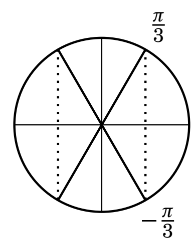
Solve

3. $4\sin^2(\theta)=3$.

Divide both sides by 4 to get $\sin^2(\theta)=\frac{3}{4}$, and then take the square root of both sides to get $\sin(\theta)=\pm\sqrt{\frac{3}{4}}$, which is $\sin(\theta)=\pm\frac{\sqrt{3}}{2}$. Note that if $\theta=\frac{\pi}{3}$ or $\theta=-\frac{\pi}{3}$, then $\sin(\theta)=\pm\frac{\sqrt{3}}{2}$, so $\theta=\frac{\pi}{3}$ and $\theta=-\frac{\pi}{3}$ are two solutions. And we can add an integer multiple of π to either of these two solutions to get another value $\theta=k\pi\pm\frac{\pi}{3}$ for which $\sin(\theta)=\pm\frac{\sqrt{3}}{2}$.

Therefore the solutions are all θ for which

$\boxed{\theta=k\pi\pm\frac{\pi}{3},\text{ where } k=0,\pm1,\pm2,\pm3,\ldots.}$



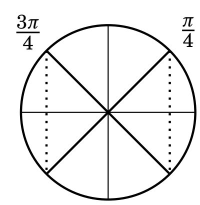
Solve

5. $2\cos^2(\theta)-1=0$.

Write this as $2\cos^2(\theta)=1$. Then divide both sides by 2 to get $\cos^2(\theta)=\frac{1}{2}$. Take the square root of both sides to get $\cos(\theta)=\pm\sqrt{\frac{1}{2}}$, which is $\cos(\theta)=\pm\frac{1}{\sqrt{2}}=\pm\frac{\sqrt{2}}{2}$. Note that if $\theta=\frac{\pi}{4}$ or $\theta=-\frac{\pi}{4}$, then $\cos(\theta)=\frac{\sqrt{2}}{2}$, so $\theta=\frac{\pi}{4}$ and $\theta=-\frac{\pi}{4}$ are two solutions. And we can add an integer multiple of π to either of these two solutions to get another value $\theta=k\pi\pm\frac{\pi}{4}$ for which $\cos(\theta)=\pm\frac{\sqrt{2}}{2}$.

Therefore the solutions are all θ for which

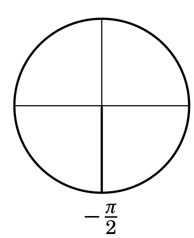
$\boxed{\theta=k\pi\pm\frac{\pi}{4},\text{ where } k=0,\pm1,\pm2,\pm3,\ldots.}$



7. Solve $2\sin(\theta)\cos(\theta)=-1$.

Using Equation (3.14), this becomes $\sin(2\theta)=-1$. From this, we see that $2\theta=-\frac{\pi}{2}+2k\pi$, where k is an integer. Dividing by 2 to isolate θ, we get

$\boxed{\theta=k\pi-\frac{\pi}{4},\text{ where } k=0,\pm1,\pm2,\pm3,\ldots.}$



9. Solve $\cos^2(\theta) - \sin^2(\theta) = \sin(2\theta)$.

Using Equation (3.15), this becomes $\cos(2\theta) = \sin(2\theta)$. The only values x for which $\cos(x) = \sin(x)$ are $x = \frac{\pi}{4} + k\pi$ (k an integer) which means $2\theta = \frac{\pi}{4} + k\pi$. Dividing by 2 to isolate θ gives the solutions as $\boxed{\theta = \frac{\pi}{8} + \frac{k\pi}{2}, \text{ where } k = 0, \pm1, \pm2, \pm3, \ldots}$

11. Solve $\sin^2(x) - 1 = \cos(x)$.

From $\sin^2(x) + \cos^2(x) = 1$, we get $\sin^2(x) - 1 = -\cos^2(x)$, so the equation becomes

$$\begin{aligned} -\cos^2(x) &= \cos(x) \\ \cos^2(x) - \cos(x) &= 0 \\ \cos(x)\big(\cos(x) - 1\big) &= 0. \end{aligned}$$

The values of x that make the first factor zero are $x = \frac{\pi}{2} + k\pi$ (where k is an integer). The values of x that make the second factor zero are $x = 2k\pi$. Thus the solutions are all values of x for which $x = \frac{\pi}{2} + k\pi$ and $x = 2k\pi$, where $k = 0, \pm1, \pm2, \pm3, \ldots$.

13. Solve $\tan^2(x) = 3$.

Take the square root of both sides to get $\tan(x) = \pm\sqrt{3}$. Note that if $x = \frac{\pi}{3}$, then $\tan(x) = \sqrt{3}$, and if $x = \frac{2\pi}{3}$, then $\tan(x) = -\sqrt{3}$. Thus $x = \frac{\pi}{3}$ and $x = \frac{2\pi}{3}$ are two solutions. And we can add an integer multiple of π to either of these two solutions to get another solution. Therefore the solutions are all x for which

$\boxed{x = \frac{\pi}{3} + k\pi \text{ or } x = \frac{2\pi}{3} + k\pi, \text{ where } k = 0, \pm1, \pm2, \pm3, \ldots}$