CHAPTER 26

Meanings of the Derivative

In Chapter 16 we introduced the most important concept in Calculus I. The **derivative** of a function $f(x)$ is another function $f'(x)$ defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - g(x)}{h} = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$

Given an input x for the derivative, the output $f'(x)$ equals the value of either one of these limits.

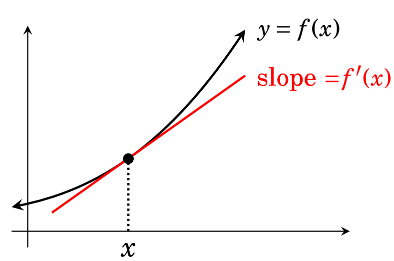
But beginning in Chapter 17 we began developing a set of derivative rules that allow us to compute $f'(x)$ without a limit *provided that* $f(x)$ *has a form that the rules apply to.* This program culminated in Chapter 25 with derivative rules for the six inverse trig functions. At that point we could quickly compute the derivative of nearly any function described by an algebraic expression, without using a limit.

This may well cause you to wonder why we need the limit definition. If we can find derivatives so effectively with the rules, who needs the limit?

There are two good reasons. First, if you encounter a function to which the rules do not apply, you may need to go back to the limit definition to work out its derivative. (This will not happen in Calculus I.) But the main reason we need the limit definition is that it gives the derivative a *meaning*. One meaning is *slope*. In fact we coded the limit formula for slope (Theorem 15.1) into the very definition of a derivative.

26.1 Slope

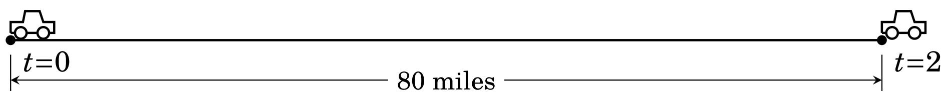
We know well at this point that given a function $f(x)$, its derivative $f'(x)$ gives the slope of the tangent to the graph of $y = f(x)$ at the point $\big(x, f(x)\big)$. This primary geometric meaning of the derivative is by now familiar and needs little review or comment.



The next derivative meaning we will explore is *velocity*.

26.2 Velocity

We will use a thought experiment to show how derivatives describe velocity. Imagine driving 80 miles on a straight highway, timing yourself with a stopwatch. At time t=0 hours you are at the starting point (below, left), and begin driving. You reach your destination at time t=2 hours (below, right).



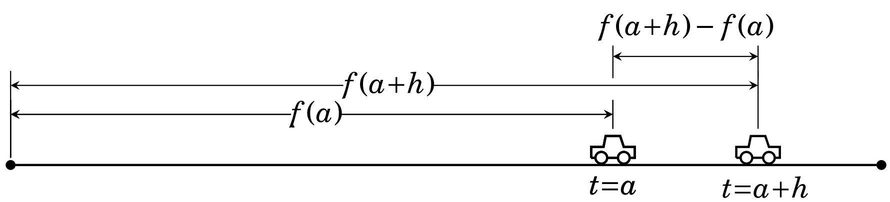
Thus your average velocity for the trip is $\frac{\text{distance traveled}}{\text{time elapsed}} = \frac{\text{80 miles}}{\text{2 hours}} = 40\,\text{mph}$. But you weren't going 40 mph the entire time. Sometimes you were going faster, sometimes slower. The question we will consider is this:

Question: *What is your exact velocity at a particular time $t = a$?*

To answer this, consider a function $f(t)$ defined as

$$f(t) = \big(\text{distance from start at time } t\big).$$

So $f(0) = 0$ and $f(2) = 80$. We don't have enough information to know other $f(a)$ values, but f is still a function whose specifics depend on how you drove.



To find the velocity at time t=a, let h be a small amount of time (say, $h = 0.1$ hours, which 6 minutes). At time t=a you've gone $f(a)$ miles. A little later, at time $t = a+h$, you've gone a total of $f(a+h)$ miles. So in h hours between times a and $a+h$ you went $f(a+h)-f(a)$ miles. Your *average* velocity between times t=a and t=$a+h$ is thus

$$v_{\text{ave}} = \frac{\text{distance traveled}}{\text{time elapsed}} = \frac{f(a+h)-f(a)}{h}.$$

This might be pretty close to your exact velocity at t=a, but a lot can happen between times a and $a+h$. You could speed up or slow down, skewing v_{ave} away from your exact velocity. For better accuracy, make h smaller, so there's less room for velocity to change in the short time from a to $a+h$. Use $h = 0.001$ hours (3.6 seconds), or $h = 0.0001$ hours (0.36 seconds). The smaller h is, the closer v_{ave} is to the exact velocity at time a. Consequently,

Answer: Velocity at time t=a is $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$ mph.

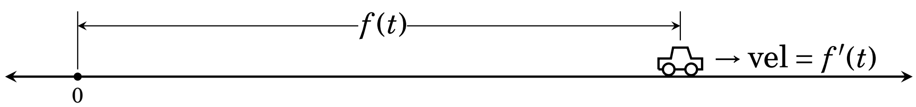
In short, your exact velocity at time t equals $f'(t)$ mph. We call $f(t)$ a **position function** because it gives the position of a moving object. The takeaway from our example is: **velocity is the derivative of position**.

Problems involving motion on a line are often modeled with the object on a number line and $f(t)$ being its location on the line at time t. Thus $f(t)$ can be positive or negative depending on whether the object is to the right or left of 0 at time t. (Also, the object doesn't necessarily have to be at the origin at time t=0.) With this convention, let's summarize our findings.

Fact 26.1 Velocity of a Moving Object:

Consider an object moving on the number line.

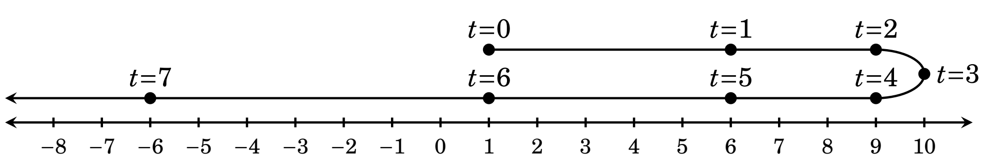
The **position function** $f(t)$ gives the object's location at time t.
The **velocity** of the object at time t is $f'(t)$.



Example 26.1 In this problem the number line is marked in meters, so any point x on the line is x meters from 0. An object moving on the number line has location $x = f(t) = 1+6t-t^2$ meters at time t seconds. Its velocity at time t is thus $f'(t) = 6-2t$ meters per second. The chart below indicates the object's position and velocity at select times.

t	(seconds)	0	1	2	3	4	5	6	7	8	⋯
$f(t) = 1+6x-x^2$	(meters)	1	6	9	10	9	6	1	−6	−15	⋯
$f'(t) = 6-2x$	(meters/sec)	6	4	2	0	−2	−4	−6	−8	−10	⋯

We tally this trajectory visually on the number line (below). At time t=0 the object is at 1, moving to the right. Three seconds later it has reached 10. There it stops moving right and begins to move left. At t=7 it has reached the point −6 and continues moving left forever.



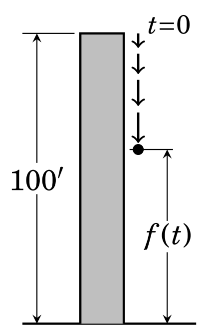
Initially, at time t=0, the object's velocity is 6 meters/sec. After that the velocity decreases steadily by 2 meters per second each second. By time t=3, it has slowed down to 0 meters per second. (This is the instant that it stops moving right and begins moving left.) ✍

In Example 26.1 the velocity $f'(x)$ is positive when the object is moving right and negative when it is moving left. This is because when an object on a number line is moving to the right, its position is increasing; velocity is then positive because position is changing at a *positive* number of meters per second (distance units per time unit). If the object is moving left, then its position is *decreasing*; velocity is then negative because position is changing at a *negative* number of distance units per time unit.

Similarly, for vertical motion, where the number line is oriented vertically (with positive pointing up), velocity is positive when the object is moving up, and negative when the object is moving down.

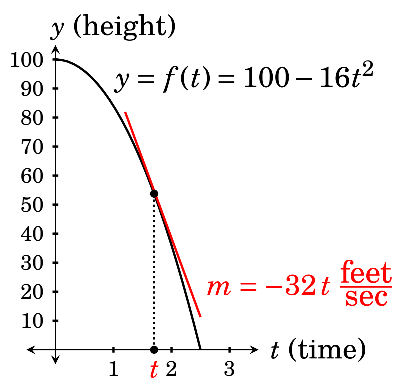
Example 26.2 A ball is dropped off a 100-foot tower at time $t=0$ seconds. A formula from physics states that it has height of $f(t) = 100 - 16t^2$ feet at time t. Find the ball's velocity at the instant it strikes ground.

Solution The ball's velocity at time t is $f'(t) = -32t$ feet per second. To find the velocity upon impact, we will find the exact time the ball hits ground and plug that time into $f'(t)$.



The ball hits ground at the instant its height is 0, that is, when $f(t) = 0$. So to find the time of impact we solve

$$\begin{aligned} 100 - 16t^2 &= 0 \\ 4\left(25 - 4t^2\right) &= 0 \\ 4(5-2t)(5+2t) &= 0 \end{aligned}$$



The solutions are $t = \pm 5/2$. The ball hits ground *after* $t=0$, so we use $t = 5/2$ seconds. So velocity on impact is

$$\boxed{f'(5/2) = -32 \cdot 5/2 = -80 \text{ feet/sec}}$$

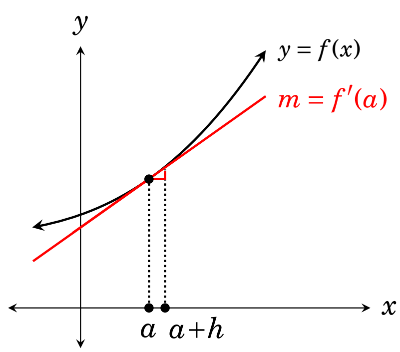
(Negative since the ball moves down.)

Notice that $f'(t) = -32t$ gives both the ball's velocity at time t **and** the slope of the tangent to $y = f(t)$ at t. In general, velocity at time t equals the slope of the tangent to $y = f(t)$ at t. (See the graph above.) ✍

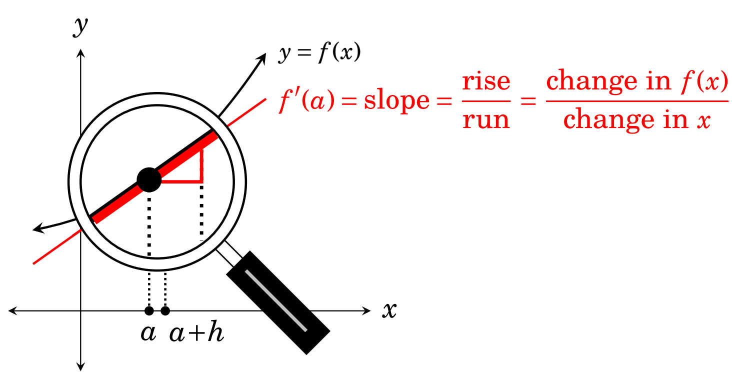
26.3 Rate of Change

We have just seen that if $f(t)$ gives the position of an object at time t, then $f'(t)$ is the object's velocity at time t. Velocity is the rate of change of distance traveled (position), so this suggests that the derivative of a function gives the rate of change of the function. We now explore this important connection.

Suppose $f(x)$ is some quantity that depends on a variable x, so changing x changes $f(x)$. Consider this at a particular value $x=a$, indicated below. Increasing x from a to $a+h$, causes $y=f(x)$ to go from $f(a)$ to $f(a+h)$. For small h, the ratio $\frac{\text{change in } f(x)}{\text{change in } x} = \frac{f(a+h)-f(a)}{h}$ is very close to the ratio $m = \frac{\text{rise}}{\text{run}} = f'(a)$ for the tangent to $y = f(x)$ at a.



Look at $(a, f(a))$ with a magnifying glass powerful enough that the graph and the tangent are indistinguishable (below). Make the increment h tiny enough that the rise/run triangle fits into the field of vision.

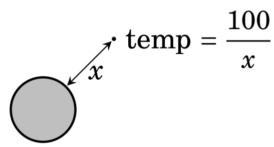


Then $f'(a) = \frac{\text{change in } f(x)}{\text{change in } x}$. This ratio—*change in f per change in x*—is the rate of change of $f(x)$ with respect to x (when $x=a$).

Conclusion: $f'(a)$ is the rate of change of $f(x)$ with respect to x, at $x=a$.

> **Fact 26.2** Suppose $f(x)$ is a quantity that depends on x. Then
>
> $f'(a)$ is the rate of change of $f(x)$ with respect to x, at $x{=}a$.
>
> We can say this more succinctly as
>
> $f'(x)$ is the rate of change of $f(x)$ with respect to x, at x

Example 26.3 Suppose that the temperature x miles above a planet's surface is given by the function $f(x) = \frac{100}{x}$ degrees celsius. So, for instance, one mile above the surface the temperature is $f(1) = \frac{100}{1} = 100$ degrees. Two miles above the surface the temperature is $f(2) = \frac{100}{2} = 50$ degrees.



By familiar rules, the derivative is $f'(x) = -\frac{100}{x^2}$. By Fact 26.2,

$$f'(x) \;=\; -\frac{100}{x^2} \;=\; \left(\begin{array}{l}\text{rate of change in temperature } f(x)\\ \text{with respect to height } x\text{, at } x\end{array}\right).$$

To illustrate this, plug a value—say $x{=}5$ miles—into the derivative.

$$f'(5) \;=\; -\frac{100}{5^2} = -4 \text{ degrees per mile}$$

This means that at five miles above the surface, temperature is *decreasing* at a rate of 4 degrees per mile. At this rate, increasing your height by one mile will decrease the temperature by 4 degrees. Next, look at height $x = 10$:

$$f'(10) \;=\; -\frac{100}{10^2} = -1 \text{ degrees per mile}$$

This means that when you are 10 miles above the surface, temperature is *decreasing* at a rate of 1 degree per mile. At this rate, increasing your height by one mile will decrease the temperature by 1 degree. However, this does *not* mean that temperature will necessarily decrease by 1 degree if you go up one mile. The value $f'(10) = -1$ is the *instantaneous* rate of change in temperature at $x = 10$, but the rate of change $f'(x)$ changes with x, so going up will change it. It's like if you are driving 60 mph one instant. It's unlikely you will travel exactly 60 miles in the next hour, because your velocity may deviate from 60 mph during that hour. ✍

Example 26.4 In economics, the cost of producing x units of a product is modeled by a cost function $C(x)$. The so-called **marginal cost** is the derivative $C'(x)$. By Fact 26.2, marginal cost $C'(x)$ is the rate of change of cost with respect to production x.

For example, say you produce 1000 units of a product. In so doing you incur a cost of $C(1000)$ dollars. At this level of production, the marginal cost is $C'(1000)$ dollars per unit. So if you increase the level of production by one unit, expect the cost to increase by $C'(1000)$. ✍

Example 26.5 Imagine that a perfectly spherical balloon is being inflated. As this happens, its radius r (centimeters) and volume V (cubic centimeters) both increase. Recall that the volume of a sphere is

$$V = \frac{4}{3}\pi r^3.$$

Thus volume is a function of radius r, and the derivative of this function is

$$\frac{dV}{dr} = 4\pi r^2.$$

By Fact 26.2, the derivative $\frac{dV}{dr}$ is the rate of change of volume V with respect to radius r. For example, if $r = 10$, then

$$\left.\frac{dV}{dr}\right|_{r=10} = 4\pi 10^2 = 400\pi.$$

This means that when the radius is 10 centimeters, volume is increasing at a rate of 400π cubic centimeters per centimeter change in r. (If r increases from 10 centimeters to 11 centimeters, volume will increase by about 400π cubic centimeters.) ✍

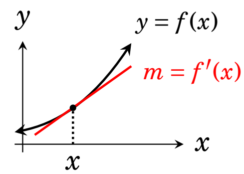
Example 26.6 Suppose $g(x)$ equals the number of gallons of fuel in your tank when you have driven your car x miles on a particular trip. Explain in practical terms what the statement $g'(100) = -0.03$ means.

Solution The derivative $g'(x)$ is the rate of change (in gallons per mile) of the number of gallons in your tank, with respect to x (distance traveled). The statement $g'(100) = -0.03$ thus means that when you have reached 100 miles traveled, the fuel your tank is changing at a rate of -0.03 gallons per mile. In other words, between the 100th and 101st mile, expect to use 0.03 gallons of fuel. ✍

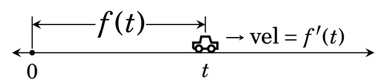
Here is a summary of the main interpretations of the derivative.

Let f be a function. Its derivative f' has the following meanings.

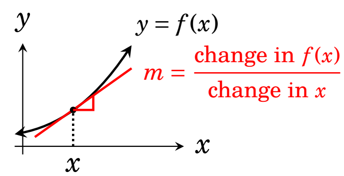
- $f'(x)$ = **slope** of tangent to $y{=}f(x)$ at $(x, f(x))$



- $f'(t) = \left(\begin{array}{l}\textbf{velocity} \text{ at time } t \text{ of object} \\ \text{whose position at time } t \text{ is } f(t)\end{array}\right)$



- $f'(x) = \left(\begin{array}{l}\textbf{rate of change} \text{ of quantity } f(x) \\ \text{with respect to } x\text{, at } x\end{array}\right)$



26.4 More on Motion: Acceleration

We now further develop the theme of motion on a line, which was laid out in Section 26.2. There we saw that if an object's location on a line at time t is given by a function $f(t)$, then $f'(t)$ is its velocity at time t. This section explains how the *second derivative* $f''(t)$ gives the object's *acceleration*.

In Section 26.2, we used f to denote a moving object's position function. Actually, in physics it is conventional to denote a position functions with the letter s. With this convention we restate Fact 26.1 as follows:

> If an object moving on the number line has position $s(t)$ at time t, then its velocity at time t is $v(t) = s'(t)$.

For example, suppose an object moving on a line is $s(t) = 1+6t-t^2$ feet from its starting point at time t seconds. Its velocity at time t is thus $s'(t) = 6-2t$ feet per second. The chart below indicates the object's position and velocity at select times.

t	(seconds)	0	1	2	3	4	5	6	7	8 ···
$s(t) = 1{+}6x{-}x^2$	(feet)	1	6	9	10	9	6	1	−6	−15 ···
$v(t) = s'(t) = 6-2x$	(feet/sec)	6	4	2	0	−2	−4	−6	−8	−10 ···

Notice how the object's velocity *decreases by* 2 *feet per second each second*. We could say that velocity changes at a rate of −2 feet per second, per second. Or, *the rate of change of velocity is* −2 *feet per second per second.*

This makes sense because from Section 26.3 we know that the derivative of a function gives its rate of change. In this sense, the rate of change of velocity $v(t) = 6 - 2t$ is $v'(t) = -2$, and this agrees with our observation that velocity is changing at a rate of -2 feet per second per second.

There is a name for the derivative of velocity. It is called **acceleration**. Thus the object above has an acceleration of -2 feet per second per second.

In general we denote acceleration as $a(t) = v'(t) = s''(t)$. Acceleration is the first derivative of velocity and the second derivative of position. Acceleration is the rate of change of velocity, so it is measured in distance units per time unit per time unit (feet/second/second, or meters/second/second, or miles/hour/hour, etc.). Here is a summary of motion on a line.

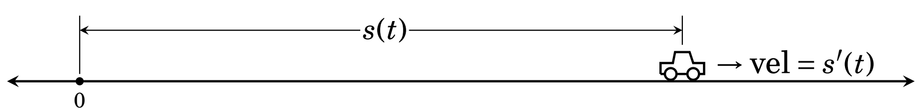
Fact 26.3 Motion on a Straight Line:

Suppose an object moving on the number line has position $s(t)$ at time t.

The object's **velocity** at time t is $v(t) = s'(t)$.

The object's **acceleration** at time t is $a(t) = v'(t)$.

The object's **speed** at time t is $|v(t)|$.



The object's *speed* is the absolute value of its velocity. Recall that *velocity* can be either positive or negative, depending on whether the object is moving right (or up) or left (or down). By contrast speed is positive. Think of it as being measured by the speedometer of a moving car. Speed is positive whether the car is moving left or right.

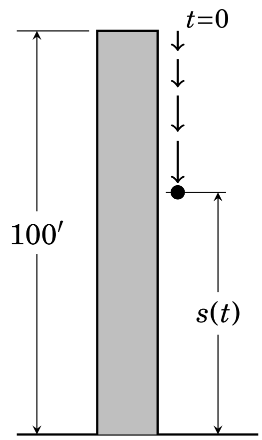
Speed and velocity are both measured in distance units per time unit (for example, miles per hour, feet per second, or meters per second).

Acceleration is the rate of change of velocity. If your acceleration is 3 feet per second per second, then your velocity is increasing at a rate of 3 feet per second, each second. However fast you are moving now, in one second you will gain another 3 feet per second in velocity.

Our next example is a continuation of Example 26.2

Example 26.7 A ball is dropped off a 100-foot tower at time t=0 seconds. A formula from physics states that it has height of $s(t) = 100 - 16t^2$ feet at time t. Find and interpret the ball's velocity and acceleration.

Solution In Example 26.2 we found that the ball strikes ground at time $t = 2.5$ seconds. We also computed its velocity at time t as $v(t) = s'(t) = -32t$ feet per second.



Its acceleration at time t is $a(t) = v'(t) = -32$ feet per second per second. Notice that acceleration is constant. This means that the ball's velocity is *decreasing* at a constant rate of 32 feet per second, each second.

This is supported in the table below, which tallies position, velocity and acceleration at various times. Notice that whatever the velocity is at any time t, one second later it has decreased by exactly 32 feet per second. So velocity is indeed decreasing at a rate of 32 feet per second per second.

t	(seconds)	0	0.5	1	1.5	2	2.5
$s(t) = 100 - 16x^2$	(feet)	100	96	84	64	36	0
$v(t) = s'(t) = -32t$	(feet/sec)	0	−16	−32	−48	−64	−80
$a(t) = v'(t) = -32$	(feet/sec/sec)	−32	−32	−32	−32	−32	−32

The constant acceleration of −32 feet per second is caused by gravity. ✍

Example 26.8 An object moves on a straight line in such a way that its distance from its starting point at time t seconds is $s(t) = 4\sqrt{t}^5$ feet. What is its velocity is when its acceleration is 30 feet per second per second?

Solution As $s(t) = 4t^{5/2}$, the object's velocity is $v(t) = 4 \cdot \frac{5}{2} t^{5/2-1} = 10t^{3/2} = 10\sqrt{t}^3$. Its acceleration is $a(t) = v'(t) = 10 \cdot \frac{3}{2} t^{3/2-1} = 15t^{1/2} = 15\sqrt{t}$. By just looking at this you can see that $t = 4$ makes the acceleration 30 feet per second per second. At this time the velocity is $v(4) = 10\sqrt{4}^3 = 80$ feet per second. ✍

Exercises for Chapter 26

1. An object moving on a straight line is $s(t) = t^2 + \sqrt{t+1} - 1$ feet from its starting point at time t seconds. Find its velocity at time $t = 8$ seconds.

2. An object moving on a straight line is $s(t) = \sqrt{t} + t^2 + 3$ feet from its starting point at time t seconds. Find its velocity at time $t = 4$ seconds.

3. An object moves on a straight line in such a way that its distance from its starting point at time t seconds is $s(t) = 3\sqrt[3]{t}^4 + 4t$ feet. How far away from the starting point is it when its velocity is 12 feet per second?

4. This problem concerns a rock that is thrown straight up in the air at time $t = 0$. At time t (in seconds) it has a height of $s(t) = 64t - 16t^2$ feet. When does the rock hit the ground? What is its velocity when it hits the ground?

5. This problem concerns a rock that is thrown off a tower at time $t = 0$. At time t (in seconds) it has a height of $s(t) = 48 + 32t - 16t^2$ feet. When does the rock hit the ground? What is its velocity when it hits the ground?

6. An object moves on a straight line in such a way that its distance from a fixed point at time t is $s(t) = t^3 - 9t^2 + 15t + 4$. Find the times t at which its velocity is 0. When is the object moving left? When is it moving right?

7. An object moves on a straight line in such a way that its distance from a fixed point at time t is $f(t) = t^3 - 15t^2 + 48t$. Find the times t at which its velocity is 0. When is the object moving left? When is it moving right?

8. An object moving on a straight line is $s(t) = \frac{t}{t^2+1}$ feet from its starting position at time t seconds. What is its acceleration when its velocity is zero feet per second?

9. An object moving on a straight line is $s(t) = t^3 - 3t^2$ feet from its starting point at time t seconds. What is the object's velocity at time t? What is its acceleration at time t? Find its acceleration when its velocity is -3 feet per second.

10. An object moving on the number line has a position of $s(t) = \tan^{-1}$ at time t. Find its velocity at time t. Find its acceleration at time t. For which times t is the object moving to the right? For which t is it speeding up? For which t is it slowing down?

11. An object moving on a straight line is $s(t) = 2 + t + t^3$ feet from its starting point at time t seconds. What is the object's velocity at time t? What is its acceleration at time t? Find its velocity when its acceleration is 12 feet per second per second.

12. Given that the surface area of a sphere of radius r inches is $S = 4\pi r^2$ square inches, find and interpret the rate of change of surface area S with respect to radius r.

13. Given that the area of a circle of radius r inches is $A = \pi r^2$ square inches, find and interpret the rate of change of area A with respect to radius r.

14. Suppose the temperature in a kiln at time t (in minutes) is $203+6\sqrt{t}$ degrees F. What it the rate of change of the temperature at the time $t=9$?

15. Suppose you begin a road trip at time $t=0$ and $f(t)$ gives your distance you have traveled after t hours. Your average velocity for the trip at any time t is distance traveled divided by time elapsed, which is $A(t)=f(t)/t$ mph. Suppose at time $t=2$ you have gone 100 miles, and are moving at a velocity of 20 miles per hour. What is the rate of change of your average velocity $A(t)$ at time $t=2$?

16. Is it possible for the population of a city to decrease while the rate of change of population increases?

17. Is it possible for the national debt to increase while its rate of change decreases?

18. Consider the function $g(v)$, where $g(v)$ equals your car's gas mileage when you are driving v miles per hour. Suppose $g'(60)=0.75$. If you are driving 60 mph and you want to improve your gas mileage, should you speed up or slow down?

19. Consider the function $h(x)$, where $h(x)$ equals the elevation (in feet above sea level) x miles due west of your present location. Suppose $h'(75)=5$. Explain what this means.

20. Consider the function $T(x)$, where $T(x)$ equals the the temperature of the atmosphere (in degrees C) x meters above your present location. Suppose $T'(900)=-2.5$. Explain what this means.

21. Suppose it costs $C(x)$ dollars to build a transmitting tower that is x meters high. Explain the meaning of $C'(x)$. Explain in simple terms the meaning of the statement $C'(100)=1000$.

22. Suppose $f(x)$ is the number of liters of fuel in a rocket when it is x miles above the Earth's surface. Explain in simple terms the meaning of the statement $f'(20)=-8$.

23. It takes a certain competitive eater $f(x)$ minutes to eat x hotdogs. (It's understood that x need not be an integer. For instance, $x=0.6$ is 60% of a hotdog.) Explain in simple terms (that the eater would understand) what $f'(x)$ means. It is somehow determined that $f'(30)=4$. What does that mean?

Exercise Solutions for Chapter 26

1. An object moving on a straight line is $s(t)=t^2+\sqrt{t+1}-1$ feet from its starting point at time t seconds. Find its velocity at time $t=8$ seconds.

Velocity at time t is $v(t)=s'(t)=2t+\frac{1}{2\sqrt{t+1}}$ feet per second. Therefore the velocity at time $t=8$ is $v(8)=2\cdot 8+\frac{1}{2\sqrt{8+1}}=16+\frac{1}{6}=\frac{97}{6}=16.1\bar{6}$ feet per second.

3. An object moves on a straight line in such a way that its distance from its starting point at time t seconds is $s(t)=3\sqrt[3]{t^4}+4t$ feet. How far away from the starting point is it when its velocity is 12 feet per second?

Velocity at time t is $s'(t) = 3\cdot\frac{4}{3}t^{4/3-1}+4 = 4t^{1/3}+4 = 4\sqrt[3]{t}+4$. To find the time at which the velocity is 12 feet per second, we need to solve

$$\begin{aligned} v(t) &= 12 \\ 4\sqrt[3]{t}+4 &= 12 \\ \sqrt[3]{t}+1 &= 3 \\ \sqrt[3]{t} &= 2 \\ t &= 2^3 = 8. \end{aligned}$$

So velocity 12 feet per second when $t = 8$ seconds. At this time the object's position is $s(8) = 3\sqrt[3]{8}^4 + 4\cdot 8 = 3\cdot 2^4 + 4\cdot 8 = 80$ feet.

5. This problem concerns a rock that is thrown off a tower at time $t = 0$. At time t (in seconds) it has a height of $s(t) = 48+32t-16t^2$ feet. When does the rock hit the ground? What is its velocity when it hits the ground?

The rock hits ground when $s(t) = 48+32t-16t^2 = 0$ feet, so we need to solve

$$\begin{aligned} 48+32t-16t^2 &= 0 \\ 3+2t-t^2 &= 0 \\ (3-t)(1+t) &= 0. \end{aligned}$$

The solutions are $t = 3$ and $t = -1$. But t must be positive in this problem, so the rock strikes ground at time $t = 3$. Velocity is at time t is $v(t) = s'(t) = 32-32t$, so velocity on impact is $v(3) = 32-32\cdot 3 = -64$ feet per second.

7. An object moves on a straight line in such a way that its distance from a fixed point at time t is $f(t) = t^3-15t^2+48t$. Find the times t at which its velocity is 0. When is the object moving left? When is it moving right?

The object's velocity at time t is $f'(t) = 3t^2-30t+48 = 3(t^2-10t+16) = 3(t-2)(t-8)$. Thus the velocity is zero at times $t = 2$ and $t = 8$. For times t between 2 and 8 $(2 < t < 8)$, the factor $(t-2)$ is positive and the factor $(t-8)$ is negative, so the velocity $f'(t) = 3(t-2)(t-8)$ is negative; hence the object is moving left when $2 < t < 8$. For other values of t, factors $(t-2)$ and $(t-8)$ are either both positive or both negative, so, $f'(t) = 3(t-2)(t-8) > 0$, meaning the object is moving right.

9. An object moving on a straight line is $s(t) = t^3-3t^2$ feet from its starting point at time t seconds. What is the object's velocity at time t? What is its acceleration at time t? Find its acceleration when its velocity is -3 feet per second.

Velocity at time t is $v(t) = s'(t) = 3t^2-6t$ feet per second.
Acceleration at time t is $a(t) = v'(t) = 6t-6$ feet per second per second.

To find the time that the velocity is −3 feet per second, we solve $v(t) = 0$.

$$\begin{aligned} 3t^2 - 6t &= -3 \\ 3t^2 - 6t + 3 &= 0 \\ t^2 - 2t + 1 &= 0 \\ (t-1)(t-1) &= 0 \end{aligned}$$

Thus velocity is −3 feet per second when $t = 1$. At this time the acceleration is $a(-3) = 6(-3) - 6 = -24$ feet per second per second.

11. An object moving on a straight line is $s(t) = 2 + t + t^3$ feet from its starting point at time t seconds. What is the object's velocity at time t? What is its acceleration at time t? Find its velocity when its acceleration is 12 feet per second per second.

The velocity at time t is $v(t) = s'(t) = 1 + 3t^2$ feet per second. The acceleration at time t is $a(t) = v'(t) = 6t$ feet per second per second.

Thus at time $t = 2$ seconds, the acceleration is $a(t) = 12$ feet per second per second. At this time the velocity is $v(2) = 1 + 3 \cdot 2^2 = 13$ feet per second.

13. Given that the area of a circle of radius r inches is $A = \pi r^2$ square inches, find and interpret the rate of change of area A with respect to radius r.

The rate of change is $\frac{dA}{dt} = 2\pi r$ square inches per inch. That is, the instantaneous rate of change of area is $2\pi r$ square inches of area per inch of radius. So if (say) the radius is $r = 3$ inches, and increasing, the area is increasing at a rate of $2\pi r = 6\pi$ square inches per inch of r. If s $r = 4$ inches, and increasing, the area is increasing at a rate of $2\pi r = 8\pi$ square inches per inch of r, etc.

15. Suppose you begin a road trip at time $t = 0$ and $f(t)$ gives your distance you have traveled after t hours. Your average velocity for the trip at any time t is distance traveled divided by time elapsed, which is $A(t) = f(t)/t$ mph. Suppose at time $t = 2$ you have gone 100 miles, and are moving at a velocity of 20 miles per hour. What is the rate of change of your average velocity $A(t)$ at time $t = 2$?

The rate of change of $A(t)$ is $A'(t) = \dfrac{f'(t) \cdot t - f(t) \cdot 1}{t^2}$ (by the quotient rule). From the information given, at time $t = 2$ you've gone $f(2) = 100$ miles and are are moving at a velocity of $f'(2) = 20$ mph. Thus $A'(2) = \dfrac{f'(2) \cdot 2 - f(2) \cdot 1}{2^2} = \dfrac{20 \cdot 2 - 100}{4} = -15$ miles per hour per hour. At this rate your average velocity for the trip is *decreasing* at a rate of 15 mph per hour.

17. Is it possible for the national debt to increase while its rate of change decreases?

Yes, this is possible. For example, suppose that at time t (in years since 2020) the national debt is $y = \ln(t)$ trillion dollars, which is unrealistic, but possible (or at least conceivable). The function $y = \ln(t)$ increases as t increases. But its rate of change is $\frac{dy}{dt} = \frac{1}{t}$ trillion dollars per year, and this *decreases* as time t increases.

19. Consider the function $h(x)$, where $h(x)$ equals the elevation (in feet above sea level) x miles due west of your present location. Suppose $h'(75) = 5$. Explain what this means.

Note that $h'(x)$ is the rate of change of elevation (in feet per mile) x miles due west of your present location. Thus $h'(75) = 5$ means that 75 miles due west of your present location, elevation is increasing at a rate of 5 feet per mile.

21. Suppose it costs $C(x)$ dollars to build a transmitting tower that is x meters high. Explain the meaning of $C'(x)$. Explain in simple terms the meaning of the statement $C'(100) = 1000$.

$C'(x)$ is the rate of change, in dollars per meter (at height x) of the cost of building the tower. In other words, if the tower is currently x meters high, then the cost of increasing its height is changing at a rate of $C'(x)$ dollars per meter. If the tower is x meters high, expect it to cost $C'(x)$ dollars to increase its height by one meter. Therefore $C'(100) = 1000$ means that if the tower is 100 meters high, it should cost about $1000 to increase the height to 101 meters.

23. It takes a certain competitive eater $f(x)$ minutes to eat x hotdogs. (It's understood that x need not be an integer. For instance, $x = 0.6$ is 60% of a hotdog.) Explain in simple terms (that the eater would understand) what $f'(x)$ means. It is somehow determined that $f'(30) = 4$. What does that mean?

The derivative $f'(x)$ is the rate of change (in minutes per hotdog) of the number of minutes it takes him to eat x hotdogs. In other words, if he's eaten x hotdogs, expect it to take him another $f'(x)$ minutes to eat the next one. Thus $f'(30) = 4$ means that once he's eaten 30 hotdogs he is eating at a rate of 4 minutes per hotdog. Expect it to take him 4 minutes to eat the 31st hotdog.