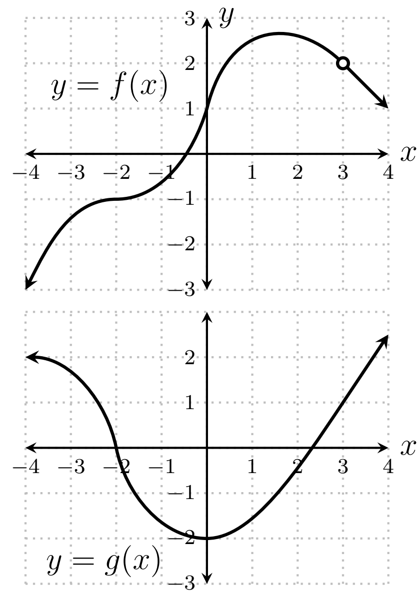
Name: ____________________ **SAMPLE MIDTERM** MATH 200, SECTIONS 7 & 8

Directions: Closed book, closed notes, no calculators. Put all phones, etc., away. You will need only a pencil or pen.

1. (6 points) Answer the questions about the functions graphed below.



(a) $\lim_{x\to 3} f(x) = \boxed{2}$

(b) $$\lim_{x\to 0} \frac{2f(x)g(x)+4f(x)}{g(x)+2} = \lim_{x\to 0} \frac{2f(x)\Big(g(x)+2\Big)}{g(x)+2} = \lim_{x\to 0} 2f(x) = 2\cdot 1 = \boxed{2}$$

(Because this limit had the form 0/0, our strategy was to factor and cancel.)

(c) $\lim_{x\to 3} g\big(x^2-6\big) = g\Big(\lim_{x\to 3}\big(x^2-6\big)\Big) = g(3) = \boxed{1}$

2. (8 points) Draw the graph of **one** function $f(x)$ meeting **all** of the following conditions.

(a) The domain of f is $(-\infty,1)\cup(1,\infty)$.

(b) The function f is continuous at all x except $x=-2$, $x=1$ and $x=4$.

(c) $\lim_{x\to 1} f(x) = -\infty$

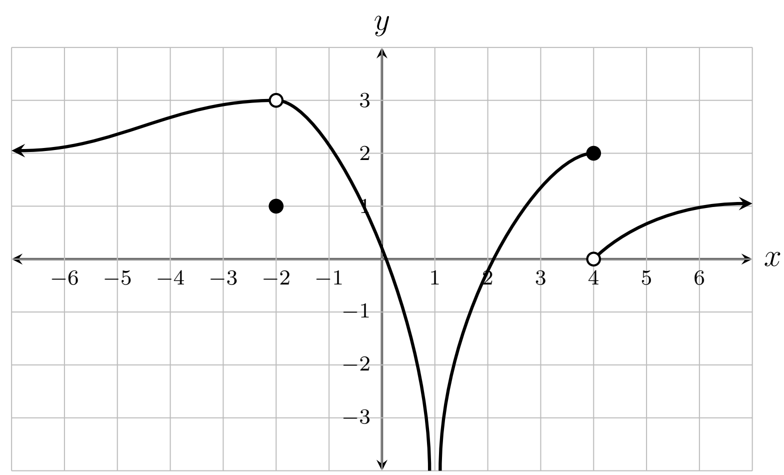
(d) $\lim_{x\to -2} f(x) = 3$

(e) $\lim_{x\to 4^-} f(x) = 2$

(f) $\lim_{x\to 4^+} f(x) = 0$

(g) $\lim_{x\to\infty} f(x) = 1$

(h) $\lim_{x\to-\infty} f(x) = 2$



3. (6 points) Find the limits

(a) $$\lim_{x\to 5}\cos\left(\frac{\pi x}{3}\right) = \cos\left(\lim_{x\to 5}\frac{\pi x}{3}\right) = \cos\left(\frac{5\pi}{3}\right) = \boxed{\frac{1}{2}}$$

(b) $$\lim_{x\to 0}\ln\left(4x+e^{x+7}\right) = \ln\left(\lim_{x\to 0}\left(4x+e^{x+7}\right)\right) = \ln\left(\left(4\cdot 0+e^{0+7}\right)\right) = \ln\left(e^7\right) = \boxed{7}$$

(c) $$\lim_{x\to\infty}\frac{4x^3-3x+10}{5x^2-6x^3} = \lim_{x\to\infty}\frac{(4x^3-3x+10)\,\frac{1}{x^3}}{(5x^2-6x^3)\,\frac{1}{x^3}} = \lim_{x\to\infty}\frac{4-\frac{3}{x^2}+\frac{10}{x^3}}{\frac{5}{x}-6} = \frac{4-0+0}{0-6} = \boxed{-\frac{2}{3}}$$

4. (8 points) Use a **limit definition** of the derivative to find the derivative of $f(x) = \sqrt{x+1}$.

$$
\begin{aligned}
f'(x) &= \lim_{z \to x} \frac{f(z) - f(x)}{z - x} \\
&= \lim_{z \to x} \frac{\sqrt{z+1} - \sqrt{x+1}}{z - x} \\
&= \lim_{z \to x} \frac{\sqrt{z+1} - \sqrt{x+1}}{z - x} \cdot \frac{\sqrt{z+1} + \sqrt{x+1}}{\sqrt{z+1} + \sqrt{x+1}} \\
&= \lim_{z \to x} \frac{\sqrt{z+1}^2 + \sqrt{z+1}\sqrt{x+1} - \sqrt{x+1}\sqrt{z+1} - \sqrt{x+1}^2}{(z-x)\left(\sqrt{z+1} + \sqrt{x+1}\right)} \\
&= \lim_{z \to x} \frac{(z+1) - (x+1)}{(z-x)\left(\sqrt{z+1} + \sqrt{x+1}\right)} \\
&= \lim_{z \to x} \frac{(z-x)}{(z-x)\left(\sqrt{z+1} + \sqrt{x+1}\right)} \\
&= \lim_{z \to x} \frac{1}{\sqrt{z+1} + \sqrt{x+1}} \\
&= \frac{1}{\sqrt{x+1} + \sqrt{x+1}} \\
&= \frac{1}{2\sqrt{x+1}}
\end{aligned}
$$

Answer: $\boxed{f'(x) = \frac{1}{2\sqrt{x+1}}}$

5. (8 points) The graph of a function $f(x)$ is sketched below.

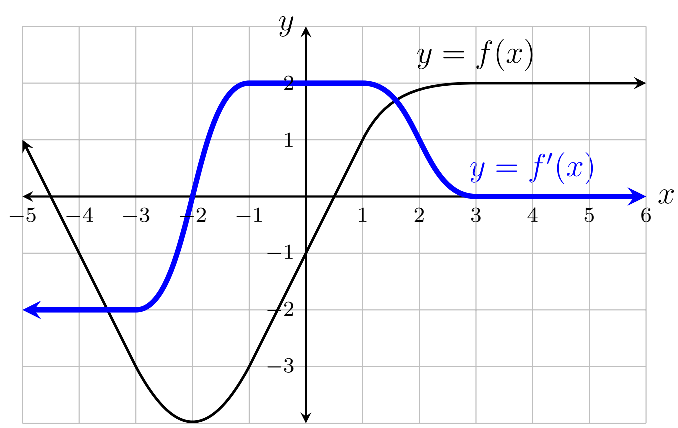
(a) Using the same coordinate axis, sketch a graph of the derivative $f'(x)$.

(b) Suppose $g(x) = \frac{1}{f(x)}$. Find $g'(0)$.

$g(x) = \left(f(x)\right)^{-1}$

$g'(x) = -\left(f(x)\right)^{-2} f'(x) = -\frac{f'(x)}{\left(f(x)\right)^2}$

$g'(0) = -\frac{f'(0)}{\left(f(0)\right)^2} = -\frac{2}{(-1)^2} = \boxed{-2}$



6. (8 points) Find all x for which the tangent to the graph of $f(x) = x^2e^x - 2$ at $\left(x, f(x)\right)$ is horizontal.

$$
\begin{aligned}
\text{We need to solve the equation} \quad f'(x) &= 0 \\
2xe^x + x^2e^x - 0 &= 0 \\
xe^x(2+x) &= 0
\end{aligned}
$$

As e^x is positive for any x, the only way this could be zero is if $x = 0$ or $x = -2$.

$\boxed{\text{Tangent is horizontal at } x = 0 \text{ or } x = -2.}$

7. (32 points) Find the derivatives of these functions. You do **not** need to simplify your answers.

(a) $f(x) = 5x^7 + 3x - \sqrt{2} \qquad f'(x) = \boxed{35x^6 + 3}$

(b) $f(x) = \sin(x)\sec(x) \qquad f'(x) = \boxed{\cos(x)\sec(x) + \sin(x)\sec(x)\tan(x)}$

(c) $f(x) = \sin\big(\sec(x)\big) \qquad f'(x) = \boxed{\cos\big(\sec(x)\big)\sec(x)\tan(x)}$

(d) $f(x) = \sin^{-1}\left(x^3 + 3x\right) \qquad f'(x) = \dfrac{1}{\sqrt{1 - \left(x^3 + 3x\right)^2}}\left(3x^2 + 3\right) = \boxed{\dfrac{3x^2 + 3}{\sqrt{1 - \left(x^3 + 3x\right)^2}}}$

(e) $f(x) = x + \dfrac{\ln(x)}{x} \qquad f'(x) = 1 + \dfrac{\frac{1}{x}x - \ln(x) \cdot 1}{x^2} = \boxed{1 + \dfrac{1 + \ln(x)}{x^2}}$

(f) $f(x) = \dfrac{1}{\sqrt{e^x + x}} = \left(e^x + x\right)^{-1/2} \qquad f'(x) = -\dfrac{1}{2}\left(e^x + x\right)^{-1/2-1}\left(e^x + 1\right) = \boxed{-\dfrac{e^x + 1}{2\sqrt{e^x + x}^3}}$

(g) $y = \cos\left(e^{x^2+x}\right) \qquad f'(x) = -\sin\left(e^{x^2+x}\right) D_x\left[e^{x^2+x}\right] = \boxed{-\sin\left(e^{x^2+x}\right)e^{x^2+x}(2x + 1)}$

(h) Given that $z = w\cos(w)$, find $\dfrac{d^2z}{dw^2}$.

$$\frac{dz}{dw} = \boxed{\cos(w) - w\sin(w)}$$

$$\frac{d^2z}{dw^2} = -\sin(w) - \sin(w) - w\cos(w) = \boxed{-2\sin(w) - w\cos(w)}$$

8. (8 points) A rocket has a height of $t+t^2$ meters t seconds after it is launched. How high is the rocket when its velocity is 101 meters per second?

The height at time t is $s(t) = t+t^2$ meters.

The velocity at time t is $v(t) = s'(t) = 1 + 2t$ meters per second.

We are interested in finding the time t at which the velocity is 101 meters per second.

For this we must solve the equation $v(t) = 101$.

$$\begin{aligned} v(t) &= 101 \\ 1 + 2t &= 101 \\ 2t &= 100 \\ t &= 50 \end{aligned}$$

Thus at time $t = 50$ seconds, the velocity is 101 meters per second.

At this time the heights is $s(50) = 50+50^2 = 50 + 2500 = \boxed{2550 \text{ meters}}$

9. (8 points) Given the equation $\ln|x+y| = xy+1$, find y'.

$$\begin{aligned} \ln|x+y| &= xy+1 \\ D_x\Big[\ln|x+y|\Big] &= D_x\Big[xy+1\Big] \\ \frac{1}{x+y}(1+y') &= 1 \cdot y + x \cdot y' + 0 \\ 1 + y' &= (x+y)(y+xy') \\ 1 + y' &= xy + x^2y' + y^2 + xyy' \\ y' - x^2y' - xyy' &= xy + y^2 - 1 \\ y'\left(1 - x^2 - xy\right) &= xy + y^2 - 1 \\ y' &= \frac{xy + y^2 - 1}{1 - x^2 - xy} \end{aligned}$$

10. (8 points) A spherical balloon is deflating in such a way that its volume is decreasing at a rate of 18 cubic feet per hour. At what rate is the radius changing when the radius is 3 feet?

Know: $\dfrac{dV}{dt} = -18$ cubic feet per hour (negative because volume is decreasing).

Want: $\dfrac{dr}{dt}$ (when $r = 3$)

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ D_t\Big[V\Big] &= D_t\left[\frac{4}{3}\pi r^3\right] \\ \frac{dV}{dt} &= \frac{4}{3}\pi 3r^2 \frac{dr}{dt} \\ \frac{dV}{dt} &= 4\pi r^2 \frac{dr}{dt} \\ \frac{dr}{dt} &= \frac{1}{4\pi r^2}\frac{dV}{dt} \\ \frac{dr}{dt} &= \frac{1}{4\pi r^2} \cdot (-18) \quad = \quad -\frac{9}{2r^2} \end{aligned}$$

Answer: $\left.\dfrac{dr}{dt}\right|_{r=3} = -\dfrac{9}{2\pi 3^2} = \boxed{-\dfrac{1}{2\pi} \text{ feet per hour}}$