Name: Richard

MIDTERM EXAM

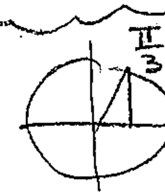
MATH 200
October 19, 2023

1. (35 pts.) Evaluate the following limits. Show steps, as appropriate.

(a) $\lim_{x\to 0} \frac{\pi \sin(x)}{3x} = \frac{\pi}{3} \lim_{x\to 0} \frac{\sin(x)}{x} = \frac{\pi}{3} \cdot 1 = \boxed{\frac{\pi}{3}}$

(b) $\lim_{x\to\infty} \frac{\sin(x)}{x} = \boxed{0} \leftarrow$ The denominator goes to ∞, but numerator oscilates between -1 and 1

(c) $\lim_{x\to\pi/3} \frac{\sin(x)}{x} = \frac{\sin(\pi/3)}{\pi/3} = \frac{\frac{\sqrt{3}}{2}}{\frac{\pi}{3}} = \boxed{\frac{3\sqrt{3}}{2\pi}}$



(d) $\lim_{x\to-\infty} \frac{x^2-3x-10}{x^2-8x+15} = \lim_{x\to-\infty} \frac{x^2-3x-10}{x^2-8x+15} \cdot \frac{1/x^2}{1/x^2}$

$= \lim_{x\to-\infty} \frac{1 - 3/x - 10/x^2}{1 - 8/x - 15/x^2} = \frac{1-0-0}{1-0-0} = \frac{1}{1} = \boxed{1}$

(e) $\lim_{x\to 5} \frac{x^2-3x-10}{x^2-8x+15} = \lim_{x\to 5} \frac{\cancel{(x-5)}(x+2)}{\cancel{(x-5)}(x-3)} = \lim_{x\to 5} \frac{x+2}{x-3} = \frac{5+2}{5-3} = \boxed{\frac{7}{2}}$

getting $\frac{0}{0}$

(f) $\lim_{x\to 0} \frac{(x-3)\sin(x)}{2x^2-6x} = \lim_{x\to 0} \frac{\cancel{(x-3)}\sin(x)}{2x\cancel{(x-3)}} = \lim_{x\to 0} \frac{\sin(x)}{2x}$

getting $\frac{0}{0}$

$= \frac{1}{2} \lim_{x\to 0} \frac{\sin(x)}{x} = \frac{1}{2} \cdot 1 = \boxed{\frac{1}{2}}$

(g) $\lim_{h\to 0} \frac{\sqrt{6+h}-\sqrt{6}}{h} = \lim_{h\to 0} \frac{\sqrt{6+h}-\sqrt{6}}{h} \cdot \frac{\sqrt{6+h}+\sqrt{6}}{\sqrt{6+h}+\sqrt{6}}$

$= \lim_{h\to 0} \frac{\sqrt{6+h}^2 - \sqrt{6}^2}{h(\sqrt{6+h}+\sqrt{6})} = \lim_{h\to 0} \frac{6+h-6}{h(\sqrt{6+h}+\sqrt{6})} = \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{6+h}-\sqrt{6})}$

$= \lim_{h\to 0} \frac{1}{\sqrt{6+h}-\sqrt{6}} = \frac{1}{\sqrt{6+0}+\sqrt{6}} = \frac{1}{\sqrt{6}+\sqrt{6}} = \boxed{\frac{1}{2\sqrt{6}}}$

2. (5 pts) Sketch the graph of **one** function with domain $[-8, 8]$ that meets all of the following criteria.

(a) $\lim_{x\to\infty} f(x) = 1$

(b) $\lim_{x\to-\infty} f(x) = 2$

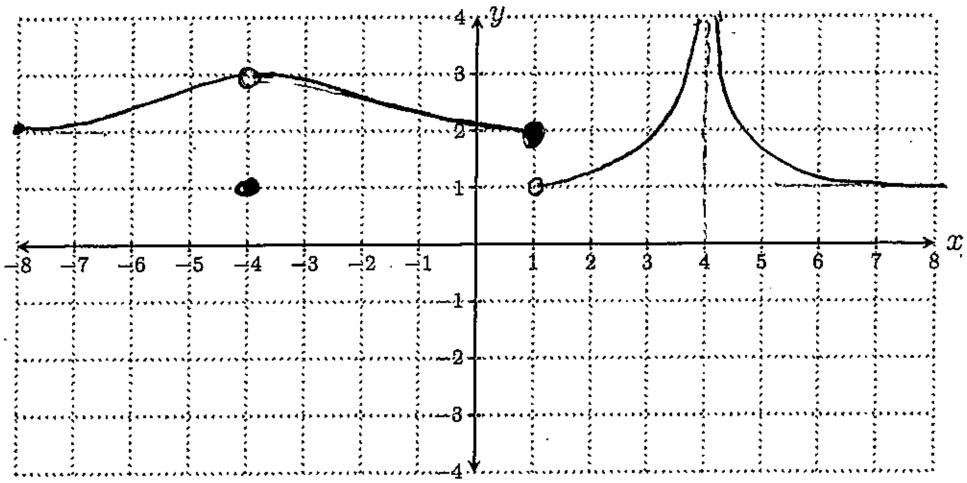
(c) $\lim_{x\to 4} f(x) = \infty$

(d) $\lim_{x\to 1^+} f(x) = 1$

(e) $\lim_{x\to 1^-} f(x) = 2$

(f) $\lim_{x\to -4} f(x) = 3$

(g) f is not continuous at $x = -4$.



3. (5 pts.) Find the following limit. Explain your reasoning.

$$f'(5) = \lim_{z\to 5} \frac{\ln(z) - \ln(5)}{z - 5} = f'(5) = \boxed{\frac{1}{5}}$$

Let $f(x) = \ln(x)$. Then $f'(x) = \frac{1}{x}$

Also $f'(x) = \lim_{z\to x} \frac{\ln(z) - \ln(x)}{z - x}$, so $f'(5) = \lim_{z\to 5} \frac{\ln(z) - \ln(5)}{z-5}$

Thus the above limit equals $f'(5) = 1/5$

4. (5 pts.) Suppose $f(x) = \frac{6}{x}$. Use a limit definition of the derivative to find $f'(x)$.

$$f'(x) = \lim_{z\to x} \frac{f(z) - f(x)}{z - x} = \lim_{z\to x} \frac{\frac{6}{z} - \frac{6}{x}}{z - x}$$

$$= \lim_{z\to x} \frac{\frac{6}{z} - \frac{6}{x}}{z - x} \cdot \frac{zx}{zx} = \lim_{z\to x} \frac{6x - 6z}{(z-x)zx}$$

$$= \lim_{z\to x} \frac{-6\cancel{(z-x)}}{\cancel{(z-x)}zx} = \lim_{z\to x} \frac{-6}{zx} = \frac{-6}{x\cdot x} = \boxed{\frac{-6}{x^2}}$$

Hence $\boxed{f'(x) = \frac{-6}{x^2}}$

5. (30 points) Find the derivatives.

(a) $\frac{d}{dx}\left[\sin^{-1}(x)\right] = \boxed{\frac{1}{\sqrt{1-x^2}}}$

(b) $\frac{d}{dx}\left[\sqrt{x^4+x^2+1}\right] = \frac{d}{dx}\left[(x^4+x^2+1)^{\frac{1}{2}}\right] = \frac{1}{2}(x^4+x^2+1)^{-\frac{1}{2}}(4x^3+2x)$

$= \frac{4x^3+2x}{2\sqrt{x^4+x^2+1}} = \boxed{\frac{2x^3+x}{\sqrt{x^4+x^2+1}}}$

(c) $\frac{d}{dx}\left[x^2\cos(x^2)\right] = \frac{d}{dx}[x^2]\cos(x^2) + x^2\frac{d}{dx}\left[\cos(x^2)\right]$

$= 2x\cos(x^2) + x^2(-\sin(x^2)2x)$

$= \boxed{2x\cos(x^2) - 2x^3\sin(x^2)}$

(d) $\frac{d}{dx}\left[\frac{e^x}{x}\right] = \frac{D_x[e^x]x - e^x D_x[x]}{x^2} = \frac{e^x x - e^x \cdot 1}{x^2}$

$= \boxed{\frac{e^x(x-1)}{x^2}}$

(e) $\frac{d}{dx}\left[\frac{1}{\sqrt{3x+1}}\right] = \frac{d}{dx}\left[(3x+1)^{-\frac{1}{2}}\right] = -\frac{1}{2}(3x+1)^{-\frac{1}{2}-1}(3)$

$= -\frac{3}{2}(3x+1)^{-3/2} = \boxed{\frac{-3}{2\sqrt{3x+1}^{\,3}}}$

(f) $\frac{d}{dx}\left[\ln(\sec(e^x))\right] = \frac{D_x[\sec(e^x)]}{\sec(e^x)} = \frac{\cancel{\sec(e^x)}\tan(e^x)e^x}{\cancel{\sec(e^x)}}$

$= \boxed{e^x\tan(e^x)}$

6. (5 pts.) Consider the equation $x^5 + 4xy^3 - 3y^5 = 2$. Use implicit differentiation to find $\frac{dy}{dx}$.

$$D_x\left[x^5 + 4xy^3 - 3y^5\right] = D_x[2] \qquad \{y = f(x)\}$$

$$5x^4 + 4y^3 + 4x\,3y^2\frac{dy}{dx} - 15y^4\frac{dy}{dx} = 0$$

$$12xy^2\frac{dy}{dx} - 15y^4\frac{dy}{dx} = -5x^4 - 4y^3$$

$$\frac{dy}{dx}\left(12xy^2 - 15y^4\right) = -5x^4 - 4y^3$$

$$\boxed{\frac{dy}{dx} = \frac{-5x^4 - 4y^3}{12xy^2 - 15y^4}}$$

7. (5 pts.) Use logarithmic differentiation to find the derivative of $f(x) = \left(\frac{1}{x}\right)^x$.

$$y = \left(\frac{1}{x}\right)^x$$

$$\ln(y) = \ln\left(\left(\frac{1}{x}\right)^x\right)$$

$$\ln(y) = x\ln\left(\frac{1}{x}\right)$$

$$\ln(y) = -x\ln(x)$$

$$D_x[\ln(y)] = D_x[-x\ln(x)]$$

$$\frac{y'}{y} = -1\cdot\ln(x) - x\frac{1}{x}$$

$$\frac{y'}{y} = -\ln(x) - 1$$

$$y' = y(-\ln(x) - 1)$$

$$\boxed{y' = -\left(\frac{1}{x}\right)^x(\ln(x) + 1)}$$

8. (10 pts.) An object is propelled straight down from atop a 160-foot-high tower at time $t = 0$ seconds. At time t seconds its height is $s(t) = 160 - 32t - 16t^2$ feet.

(a) Find the object's height when its velocity is -96 feet per second.

Know $v(t) = s'(t) = -32 - 32t$

To find when velocity is -96 ft/sec, solve

$v(t) = -96$

$-32 - 32t = -96$

$-32t = -64$

$t = -64/-32 = \boxed{2 \text{ sec.}}$ So at $t = 2$ sec, the velocity is -96 ft/sec. At this time height is

$S(2) = 160 - 32 \cdot 2 - 16 \cdot 2^2 = 160 - 64 - 64 = \boxed{32 \text{ ft}}$

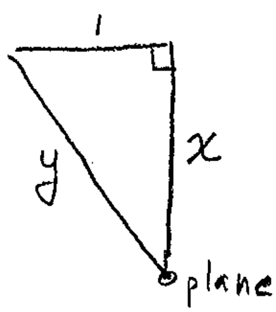
(b) What is object's acceleration when its velocity is -96 feet per second.

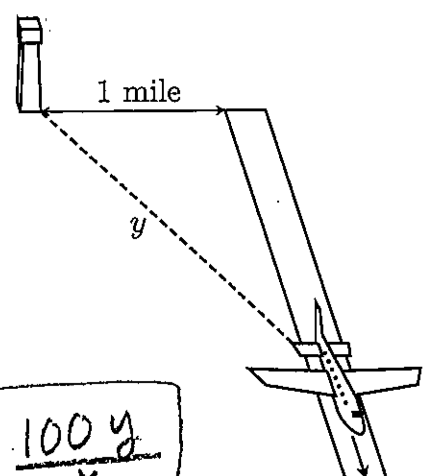
$a(t) = v'(t) = \frac{d}{dt}[-32 - 32t] = -32$ ft/sec/sec

So object's acceleration is <u>always</u> -32 ft/sec/sec

9. (**Bonus:** 5 pts.) A plane is taxing down a runway that is one mile from a tower, as shown below. When the plane is 5/3 miles from the tower, the distance y between tower and plane is increasing at a rate of 100 mph. What is the plane's velocity at this point in time?

Let x be as shown





Know $\frac{dy}{dt} = 100$ (when $y = \frac{5}{3}$)

Want $\frac{dx}{dt}$ (when $y = 5/3$)

By pythagorean theorem

$1 + x^2 = y^2$

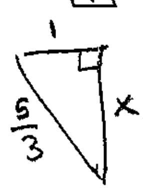
$D_t[1 + x^2] = D_t[y^2]$

$0 + 2x\frac{dx}{dt} = 2y\frac{dy}{dt}$

$\frac{dx}{dt} = \frac{2y}{2x}\frac{dy}{dt} = \frac{y}{x}100$

$\boxed{\frac{dx}{dt} = \frac{100y}{x}}$

But when $y = 5/3$,



$x = \sqrt{\left(\frac{5}{3}\right)^2 - 1^2}$

$= \sqrt{\frac{25}{9} - \frac{9}{9}} = \sqrt{\frac{16}{9}} = \frac{4}{3}$

$\boxed{\text{So } \frac{dx}{dt} = \frac{100\,5/3}{4/3} = 125 \text{ mph}}$