Name: Richard

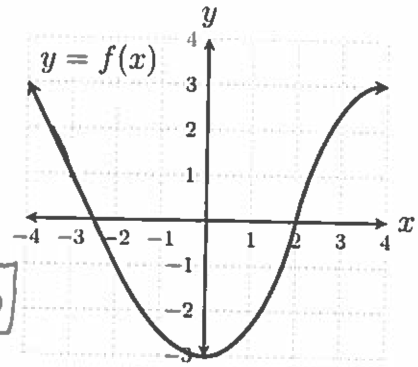
MATH 200 MIDTERM EXAM — OCT. 27, 2021

Directions: Closed book, closed notes, no calculators. Put all phones, etc., away. You will need only a pencil or pen.

1. (10 points) Answer the questions about the function f graphed below.

(a) $\lim_{h\to 0}\dfrac{f(-3+h)-f(-3)}{h} = f'(-3) = \boxed{-2}$ (Slope at $(-3, f(-3))$)

(b) $\lim_{x\to\infty} f\left(\dfrac{1}{x}\right) = f\left(\lim_{x\to\infty}\dfrac{1}{x}\right) = f(0) = \boxed{-3}$



(c) $\lim_{x\to 0}\dfrac{1}{3+f(x)} = \boxed{\infty}$

approaching 0, positive

(d) $\lim_{x\to 2}\dfrac{\sin(f(x))}{f(x)+1} = \dfrac{\lim_{x\to 2}\sin(f(x))}{\lim_{x\to 2}(f(x)+1)} = \dfrac{\sin(0)}{0+1} = \boxed{0}$

(e) $\lim_{x\to 2}\dfrac{\sin(f(x))}{f(x)} = \boxed{1}$ (because $f(x)$ approaches 0)

2. (20 points) Find the limits

(a) $\lim_{x\to 0^+}\sin^{-1}(x-1) = \sin^{-1}(0-1) = \sin^{-1}(-1) = \boxed{-\frac{\pi}{2}}$

(b) $\lim_{x\to e} 5\ln(x^3) = 5\ln\left(\lim_{x\to e} x^3\right) = 5\ln(e^3) = 5\cdot 3 = \boxed{15}$

(c) $\lim_{x\to 3}\dfrac{x-3}{x^2-7x+12} = \lim_{x\to 3}\dfrac{\cancel{x-3}}{\cancel{(x-3)}(x-4)} = \lim_{x\to 3}\dfrac{1}{x-4} = \dfrac{1}{3-4} = \boxed{-1}$

$\frac{0}{0}$, so cancel

(d) $\lim_{x\to 1}\dfrac{\frac{1}{x}-1}{x-1} = \lim_{x\to 1}\dfrac{\frac{1}{x}-1}{x-1}\cdot\dfrac{x}{x} = \lim_{x\to 1}\dfrac{1-x}{(x-1)x} = \lim_{x\to 1}\dfrac{-\cancel{(x-1)}}{\cancel{(x-1)}x}$

$= \lim_{x\to 1}\dfrac{-1}{x} = -\dfrac{1}{1} = \boxed{-1}$

$\frac{0}{0}$, so cancel

3. (7 points) Use a **limit definition** of the derivative to find the derivative of $f(x) = \sqrt{1-x}$.

$$\lim_{z\to x} \frac{f(z)-f(x)}{z-x} = \lim_{z\to x} \frac{\sqrt{1-z}-\sqrt{1-x}}{z-x}$$

$$= \lim_{z\to x} \frac{\sqrt{1-z}-\sqrt{1-x}}{z-x} \cdot \frac{\sqrt{1-z}+\sqrt{1-x}}{\sqrt{1-z}+\sqrt{1-x}}$$

$$= \lim_{z\to x} \frac{\sqrt{1-z}^{\,2}-\sqrt{1-x}^{\,2}}{(z-x)(\sqrt{1-z}+\sqrt{1-x})}$$

$$= \lim_{z\to x} \frac{1-z-(1-x)}{(z-x)(\sqrt{1-z}+\sqrt{1-x})} = \lim_{z\to x} \frac{-z+x}{(z-x)(\sqrt{1-z}+\sqrt{1-x})}$$

$$= \lim_{z\to x} \frac{-\cancel{(z-x)}}{(z-x)(\sqrt{1-z}+\sqrt{1-x})} = \lim_{z\to x} \frac{-1}{\sqrt{1-z}+\sqrt{1-x}} = \frac{-1}{\sqrt{1-x}+\sqrt{1-x}} = \boxed{\frac{-1}{2\sqrt{1-x}}}$$

4. (7 points) An object moving on a straight line is $s(t) = t^3 - 3t^2$ feet from its starting point at time t seconds. Find its acceleration when its velocity is -3 feet per second.

Velocity: $v(t) = s'(t) = 3t^2 - 6t$

Accel.: $a(t) = v'(t) = 6t - 6$

To find time t when velocity is -3, solve $v(t) = -3$

$$3t^2 - 6t = -3$$
$$3t^2 - 6t + 3 = 0$$
$$3(t^2 - 2t + 1) = 0$$
$$3(t-1)(t-1) = 0$$

So velocity is -3 when $t = 1$ sec. Then accel. is $a(1) = 6\cdot 1 - 6 = \boxed{0\ ft/sec^2}$

5. (7 points) Suppose $f(x) = x^2 + 2x^3$ and $g(x) = x^2 - 2x^3 + 48x$. Find all x for which the tangent to $y = f(x)$ at $(x, f(x))$ is parallel to the tangent to $y = g(x)$ at $(x, g(x))$.

Need to solve $f'(x) = g'(x)$

$$2x + 6x^2 = 2x - 6x^2 + 48$$
$$12x^2 - 48 = 0$$
$$12(x^2 - 4) = 0$$
$$12(x-2)(x+2) = 0$$

$x = 2$ $\quad$ $x = -2$

Answer

Slopes are the same at $x = -2$ and $x = 2$ so tangents are parallel there

6. (35 points) Find the derivatives of these functions. You do **not** need to simplify your answers.

(a) $f(x) = \frac{\sqrt{2}}{x} + \pi x$ $= \sqrt{2}x^{-1} + \pi x$ $\quad f'(x) = \sqrt{2}(-1)x^{-2} + \pi$

$= \boxed{-\frac{\sqrt{2}}{x^2} + \pi}$

(b) $f(x) = \cos(x)\sin(x)$

$f'(x) = -\sin(x)\sin(x) + \cos(x)\cos(x)$

$= \boxed{\cos^2(x) - \sin^2(x)}$

(c) $f(x) = \cos\left(\sin(x)\right)$

$f'(x) = \boxed{-\sin(\sin(x))\cos(x)}$ (chain rule)

(d) $f(x) = \tan^{-1}(-x)$ $\quad f'(x) = \frac{1}{1+(-x)^2}(-1) = \boxed{\frac{-1}{1+x^2}}$

(e) $f(x) = \ln\left(e^{x^2-3x} + x\right)$ $= \frac{D_x\left[e^{x^2-3x} + x\right]}{e^{x^2-3x} + x} = \boxed{\frac{e^{x^2-3x}(2x-3)+1}{e^{x^2-3x} + x}}$

(f) $f(x) = \frac{1}{x^2+5x-7}$ $= (x^2+5x-7)^{-1}$

$f'(x) = -(x^2+5x-7)^{-2}(2x+5) = \boxed{-\frac{2x+5}{(x^2+5x-7)^2}}$

(g) $f(x) = \sqrt{\frac{x+1}{x-1}}^3$ $= \left(\frac{x+1}{x-1}\right)^{\frac{3}{2}}$

$f'(x) = \frac{3}{2}\left(\frac{x+1}{x-1}\right)^{\frac{3}{2}-1}\frac{1\cdot(x-1)-(x+1)\cdot 1}{(x-1)^2} = \frac{3}{2}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}}\frac{-2}{(x-1)^2}$

$= \boxed{-3\sqrt{\frac{x+1}{x-1}}\,\frac{1}{(x-1)^2}}$

7. (7 points) Given the equation $\frac{x}{y} = y^5 + x$, find y'.

$$D_x\left[\frac{x}{y}\right] = D_x\left[y^5 + x\right]$$

$$\frac{1 \cdot y - xy'}{y^2} = 5y^4y' + 1$$

$$y - xy' = y^2(5y^4y' + 1)$$

$$y - xy' = 5y^6y' + y^2$$

$$y - y^2 = 5y^6y' + xy'$$

$$y - y^2 = (5y^6 + x)y'$$

$$\frac{y - y^2}{5y^6 + x} = y'$$

Answer:

$$\boxed{y' = \frac{y - y^2}{5y^6 + x}}$$

8. (7 points) Find the derivative of $f(x) = x^{\ln(x)}$. (Use logarithmic differentiation)

$$y = x^{\ln(x)}$$

$$\ln(y) = \ln(x^{\ln(x)})$$

$$\ln(y) = \ln(x)\ln(x)$$

$$D_x[\ln(y)] = D_x[\ln(x) \cdot \ln(x)]$$

$$\frac{y'}{y} = \frac{1}{x}\ln(x) + \ln(x)\frac{1}{x}$$

$$y' = y\left(2\frac{\ln(x)}{x}\right)$$

$$\boxed{y' = 2x^{\ln(x)}\frac{\ln(x)}{x}}$$