VCU
MATH 200
CALCULUS I
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TEST 2

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Score: 100

Directions. Answer the questions in the space provided. Unless noted otherwise, you must show and explain your work to receive full credit. Put your final answer in a box when appropriate.

This is a closed-book, closed-notes test. Calculators, computers, etc., are not used.
1. (20 points) Warmup: short answer.

(a) \( \cos^{-1} \left( -\frac{\sqrt{2}}{2} \right) = \frac{3\pi}{4} \)

(b) \( \tan^{-1} (-1) = -\frac{\pi}{4} \)

(c) \( \frac{d}{dx} [x^3 + 4\ln(x) + e^x] = 3x^2 + \frac{4}{x} + e^x \)

(d) \( \frac{d}{dx} [\sec(x^5)] = \sec(x^5) \tan(x^5) 5x^4 \)

(e) \( \frac{d}{dx} [(\sec(x))^5] = 5(\sec(x))^5 \sec(x) \tan(x) \)
\[= 5\sec^6(x) \tan(x) \]

(f) \( \frac{d}{dx} [e^x + \sin^{-1}(x)] = e^x + \frac{1}{\sqrt{1-x^2}} \)

(g) \( \frac{d}{dx} \left[ \frac{\sin x}{x} \right] = \frac{\cos(x)x - \sin(x)}{x^2} \)

(h) \( \lim_{x \to \infty} \tan^{-1}(x) = \frac{\pi}{2} \)

(i) \( \lim_{x \to 0^+} \ln(x) = -\infty \)

(j) \( \lim_{x \to 1} \ln(x) = 0 \)
2. (5 points) Simplify: \( \tan(\sin^{-1}(x)) = \frac{\text{OPP}}{\text{ADJ}} = \frac{x}{\sqrt{1-x^2}} \)

3. (5 points) For the function \( f(x) \) below,
   find \( \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h} = f'(3) = \left( \text{slope of tangent at } x=3 \right) = -2 \)

4. (5 points) A function \( f(x) \) is graphed below. Using the same coordinate axis, draw a rough sketch of the graph of \( f'(x) \).
5. (10 points) Find the inverse of the function \( f(x) = \frac{e^x}{1 + e^x} \).

\[
y = \frac{e^x}{1 + e^x}
\]
\[
x = \frac{e^y}{1 + e^y}
\]
\[
x(1 + e^y) = e^x
\]
\[
x + xe^y = e^y
\]
\[
x e^y - e^y = -x
\]
\[
e^y(x - 1) = -x
\]
\[
y = \ln\left(\frac{x}{x-1}\right)
\]
\[
e^y = \frac{-x}{x-1}
\]
\[
\ln(e^y) = \ln\left(\frac{-x}{x-1}\right)
\]
\[
y = \ln\left(\frac{-x}{x-1}\right)
\]
\[
f'(x) = \ln\left(\frac{-x}{x-1}\right)
\]

6. (10 points) Use logarithmic differentiation to find the derivative of \( y = x^{\cos(x)} \).

\[
\ln(y) = \ln(x^{\cos(x)})
\]
\[
\ln(y) = \cos(x) \ln(x)
\]
\[
\frac{d}{dx} [\ln(y)] = \frac{d}{dx} [\cos(x) \ln(x)]
\]
\[
\frac{1}{y} \frac{dy}{dx} = -\sin(x) \ln(x) + \cos(x) \frac{1}{x}
\]
\[
\frac{dy}{dx} = y (-\sin(x) \ln(x) + \cos(x) \frac{1}{x})
\]
\[
\frac{dy}{dx} = x^{\cos(x)} \left(\frac{\cos(x)}{x} - \sin(x) \ln(x)\right)
\]
7 (20 points) Find the following derivatives.

(a) \[ \frac{d}{dx} \left[ \sqrt{x^2+1} \right] = \frac{d}{dx} \left[ \left( x^2+1 \right)^{\frac{1}{2}} \right] = \frac{1}{2} \left( x^2+1 \right)^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{x^2+1}} \]

(b) \[ \frac{d}{dx} \left[ e^{\tan^{-1}(\pi x)} \right] = e^{\tan^{-1}(\pi x)} \cdot \frac{1}{1+(\pi x)^2} \cdot \pi = \frac{\pi e^{\tan^{-1}(\pi x)}}{1 + \pi^2 x^2} \]

(c) \[ \frac{d}{dx} \left[ \frac{\sqrt{x} \cos(x)}{x^3+1} \right] = \frac{\frac{1}{2} \sqrt{x} \cos(x) - x^2 \sin(x)}{(x^3+1)^2} \cdot \left( x^3+1 \right)^{\frac{1}{2}} - \sqrt{x} \cos(x) \cdot \frac{3x^2}{(x^3+1)^2} \]

(d) \[ \frac{d}{dx} \left[ \ln(x) e^x \right] = \frac{1}{x} e^x + \ln(x) e^x \]
8. (5 points) \[ \lim_{h \to 0} \frac{\ln(x + h) - \ln(x)}{h} = \frac{d}{dx} [\ln(x)] = \frac{1}{x} \]

- Definition of derivative
- Derivative rule

9. (10 points) Suppose \( f(x) = x^3 - x + 2 \).

Find the equation of the line tangent to the graph of \( f(x) \) at the point \((2, 8)\).

\[
f'(x) = 3x^2 - 1
\]

Slope = \( f'(2) = 3 \cdot 2^2 - 1 = 11 \)

Point-slope form

\[
y - 8 = 11(x - 2)
\]

\[
y - 8 = 11x - 22
\]

\[
y = 11x - 14
\]
10. (10 points) This question concerns the equation \( \ln(xy) = y + 1 \).

(a) Use implicit differentiation to find \( \frac{dy}{dx} \).

\[
\frac{d}{dx} \left[ \ln(xy) \right] = \frac{d}{dx} \left[ y + 1 \right]
\]

\[
\frac{1}{xy} \cdot y + xy' \frac{y}{xy} = y' + 0
\]

\[
y + xy' = xy y'
\]

\[
xy' - xy y' = -y
\]

\[
y' (x - xy) = -y
\]

\[
y' = \frac{-y}{x - xy}
\]

(b) Use your answer from part (a) to find the slope of the tangent line to the graph of \( \ln(xy) = y + 1 \) at the point \((-1, -1)\).

\[
\left. \frac{dy}{dx} \right|_{(x,y)=(-1,-1)} = \frac{-(-1)}{-1 - (-1)(-1)} = \frac{-1}{2}
\]