7. (10 pts.) Suppose \( f(x) \) is a function for which \( f'(x) = \frac{1}{x} + 3x \) and \( f(1) = 5 \). Find \( f(x) \).

\[
f(x) = \int \left( \frac{1}{x} + 3x \right) \, dx = \ln|x| + \frac{3}{2}x^2 + C
\]

Thus we just need to find \( C \).

\[
5 = f(1) = \ln 1 + \frac{3}{2} \cdot 1^2 + C
\]

\[
5 = 0 + \frac{3}{2} + C
\]

\[
C = 5 - \frac{3}{2} = \frac{10}{2} - \frac{3}{2} = \frac{7}{2}
\]

Therefore \( f(x) = \ln|x| + \frac{3}{2}x^2 + \frac{7}{2} \)

1. (32 points) Find the indefinite integrals.

(a) \[
\int (5x + 3 + x^4) \, dx = \frac{5}{2}x^2 + 3x + \frac{x^5}{5} + C
\]

(b) \[
\int \left( \frac{1}{x^2} + \sqrt{x} \right) \, dx = \int \left( x^{-2} + x^{1/2} \right) \, dx = \frac{1}{1} x^{-1+1} + \frac{1}{\frac{1}{2} + 1} x^{\frac{1}{2}+1} + C
\]

\[
= -x^{-1} + \frac{2}{3}x^{3/2} + C = -\frac{1}{x} + \frac{2}{3} \sqrt{x}^3 + C
\]

(c) \[
\int \frac{6}{\sqrt{1-x^2}} \, dx = 6 \int \frac{1}{\sqrt{1-x^2}} \, dx = 6 \sin^{-1}(x) + C
\]

(d) \[
\int 4 \sin(3x) \, dx = 4 \int \sin(3x) \, dx = 4 \left( -\frac{1}{3} \cos(3x) \right) + C
\]

\[
= -\frac{4}{3} \cos(3x) + C
\]
2. (10 pts.) Suppose you have 120 feet of fencing material to enclose two rectangular regions, as illustrated. Find the dimensions x and y that maximize the total enclosed area.

\[
\text{Area} = xy = x(40 - \frac{2}{3}x) \\
\text{Area} = A(x) = 40x - \frac{2}{3}x^2
\]

Thus we need to find the value of x that maximizes A(x) on the interval \([0, 120]\).

\[
A'(x) = 40 - \frac{4}{3}x = 0
\]

\[
x = 30
\]

\[
y = A(30)
\]

Thus area maximized when \(x = 30\), \(y = 40 - \frac{2}{3} \cdot 30 = 20\)

Answer: \(x = 30\), \(y = 20\) for maximum enclosed area

3. (10 pts.) The graph \(y = f'(x)\) of the derivative of a function \(f(x)\) is shown. Answer the questions about \(f(x)\).

(a) State the intervals on which \(f(x)\) increases.
\((-\infty, -4)\) and \((10, \infty)\) because that's where \(f(x) > 0\)

(b) State the intervals on which \(f(x)\) decreases.
\((-4, 3)\) and \((3, 10)\) because that's where \(f(x) < 0\)

(c) List all critical points of \(f(x)\).
\(-4, 3, 10\) because that's where \(f(x) = 0\)

(d) At which of these critical points is there a local maximum?
\(x = -4\) by First derivative test.

(e) State the intervals on which the function \(f(x)\) is concave up.
\((-1, 3)\) and \((7, \infty)\) because \(f''(x) > 0\) there.

\(f'(x)\) increases on these intervals, and therefore \(f''(x) > 0\) there.
4. (20 pts.) Find the limits.

(a) \[ \lim_{x \to 0} \frac{3x^2}{\cos(x) - 1} = \lim_{x \to 0} \frac{6x}{-\sin(x)} = \lim_{x \to 0} \frac{6}{-\cos(x)} = \frac{6}{-1} = -6 \]

(b) \[ \lim_{x \to 0} (1 + x)^{1/2} = \lim_{x \to 0} e^{\frac{1}{2} \ln((1 + x)^{1/2})} = \lim_{x \to 0} e^{\frac{1}{2} \ln(1 + x)} = \lim_{x \to 0} e^{\frac{1}{2} x} = \lim_{x \to 0} e^{\frac{1}{2} x} = e^{1/2} = e^{1/2} = e \]

5. (8 pts.) Is the following equation true or false?

\[ \int \frac{\sin \left( \frac{1}{x} \right)}{x^2} \, dx = \cos \left( \frac{1}{x} \right) + C \]

Explain. 

6. (10 pts.) A 13-foot ladder is leaning against a wall, as illustrated, when its base begins to slide away from the wall at a rate of 5 feet per second. At what rate is the angle \( \theta \) changing when the base is 12 feet from the wall?

\[
\begin{align*}
\text{Know} & \quad \frac{dx}{dt} = 5 \\
\text{Want} & \quad \frac{d\theta}{dt} \quad \text{when} \quad x = 12 \\
\text{From diagram,} & \quad \cos(\theta) = \frac{x}{13} \\
\frac{d}{dt} \left[ \cos(\theta) \right] & \quad = \frac{d}{dt} \left[ \frac{1}{13} x \right] \\
-\sin(\theta) \frac{d\theta}{dt} & \quad = \frac{1}{13} \frac{dx}{dt} \\
-\sin(\theta) \frac{d\theta}{dt} & \quad = \frac{1}{13} \cdot 5 \\
\frac{d\theta}{dt} & \quad = \frac{-5}{13 \sin(\theta)} \\
\end{align*}
\]

To find \( \frac{d\theta}{dt} \) we just need to find \( \sin(\theta) \) and plug it in above. When the base \( x \) is 12 the triangle looks like this:

\[ \begin{array}{c}
\text{Thus} \quad \sin(\theta) = \frac{5}{13} \quad (= \text{OPP}) \\
\text{Therefore} \quad \frac{d\theta}{dt} = \frac{-5}{13 \sin \theta} = \frac{-5}{13 \times \frac{5}{13}} = -1 \text{ rad/ sec}
\end{array} \]