7. (10 pts.) Suppose \( f(x) \) is a function for which \( f'(x) = \sqrt{x} + 2 \) and \( f(4) = 7 \). Find \( f(x) \).

\[
f'(x) = x^{\frac{1}{2}} + 2
\]

\[
f(x) = \int f'(x) \, dx = \int (x^{\frac{1}{2}} + 2) \, dx
\]

\[
= \frac{1}{\frac{1}{2}+1} x^{\frac{1}{2}+1} + C
\]

\[
= \frac{1}{\frac{3}{2}} x^{\frac{3}{2}} + 2x + C
\]

\[
f(x) = \frac{2}{3} \sqrt{x^3} + 2x + C
\]

Now we just need to find \( C \):

\[
7 = f(4) = \frac{2}{3} \sqrt{4^3} + 2 \cdot 4 + C
\]

\[
7 = \frac{16}{3} + 8 + C
\]

\[
C = -\frac{16}{3} - 8 - \frac{3}{3} = -\frac{19}{3}
\]

\[
f(x) = \frac{2}{3} \sqrt{x^3} + 2x - \frac{19}{3}
\]

1. (32 points) Find the indefinite integrals.

(a) \[
\int (x^5 + x + 1) \, dx = \frac{1}{6} x^6 + \frac{1}{2} x^2 + x + C = \frac{x^5}{6} + \frac{x^2}{2} + x + C
\]

(b) \[
\int 4e^{3x} \, dx = 4 \int e^{3x} \, dx = 4 \cdot \frac{1}{3} e^{3x} + C = \frac{4}{3} e^{3x} + C
\]

(c) \[
\int \frac{5}{1 + x^2} \, dx = \int \frac{1}{1 + x^2} \, dx = 5 \tan^{-1}(x) + C
\]

(d) \[
\int (\frac{1}{x} + \cos(x)) \, dx = \int (x^{-1} + \cos(x)) \, dx = \ln|x| + \sin(x) + C
\]
2. (10 pts.) Suppose you have 160 feet of fencing material to enclose a rectangular region. One side of the rectangle will border a building, so no fencing is required for that side. Find the dimensions x and y that maximize the fenced area.

Area = \( xy = x(80 - \frac{1}{2}x) \)

Area = \( A(x) = 80x - \frac{1}{2}x^2 \)

We need to maximize this on \([0, 160]\)

\( A'(x) = 80 - x = 0 \)

\( x = 80 \) \( \downarrow \) \{critical point\}

Thus maximum area when \( x = 80 \) and \( y = 80 - \frac{1}{2} \times 80 = 40 \).

Answer \( x = 80, y = 40 \) maximizes area.

3. (10 pts.) The graph \( y = f'(x) \) of the derivative of a function \( f(x) \) is shown. Answer the questions about \( f(x) \).

\( y = f'(x) \)

(a) State the intervals on which \( f(x) \) increases. \( \boxed{(-3, 4), (4, 11)} \) because that's where \( f'(x) > 0 \)

(b) State the intervals on which \( f(x) \) decreases. \( \boxed{(-\infty, -3), (11, \infty)} \) because that's where \( f'(x) < 0 \)

(c) List all critical points of \( f(x) \). \( \boxed{-3, 4, 11} \) because that's where \( f'(x) = 0 \).

(d) At which of these critical points is there a local maximum? \( x = 11 \) by First Derivative Test.

(e) State the intervals on which the function \( f(x) \) is concave up. \( f(x) \) is concave up on \( \boxed{(-\infty, 0) \text{ and } (4, 8)} \) because \( f''(x) > 0 \) there.
4. (20 pts.) Find the limits.

(a) \[ \lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{0 + \sin(x)}{2x} = \lim_{x \to 0} \frac{\sin(x)}{2x} = \lim_{x \to 0} \frac{\cos(x)}{2} = \frac{\cos(0)}{2} \]

Apply L'Hôpital

(b) \[ \lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{\ln(x^x)} = \lim_{x \to 0^+} e^{x \ln(x)} = \lim_{x \to 0^+} e^{\frac{1}{x}} = \lim_{x \to 0^+} e^{-\frac{1}{x^2}} = \lim_{x \to 0^+} e^{-x} = e^0 = 1 \]

5. (8 pts.) Is the following equation true or false?

\[ \int \left( \cos(x) \frac{1}{x} - \sin(x) \ln(x) \right) \, dx = \cos(x) \ln(x) + C \]

Note that \[ \frac{d}{dx} \left[ \cos(x) \ln(x) \right] = -\sin(x) \ln(x) + \cos(x) \frac{1}{x} = \cos(x) \frac{1}{x} - \sin(x) \ln(x), \]

so **YES**, it's true.

6. (10 pts.) Water flows into the conical tank (shown below) at a rate of 9 cubic feet per minute. How fast is the water level \( h \) rising when the water is 6 feet deep?

By similar triangles, \( \frac{r}{h} = \frac{5}{10} \), so \( r = \frac{1}{2} h \)

The volume of a cone of height \( h \) and radius \( r \) is \( V = \frac{1}{3} \pi r^2 h \)

\[ V = \frac{1}{3} \pi \left( \frac{h}{2} \right)^2 h \]

\[ V = \frac{\pi}{12} h^3 \]

\[ \frac{d}{dt} [V] = \frac{d}{dt} \left[ \frac{\pi}{12} h^3 \right] \]

\[ \frac{dV}{dt} = \frac{\pi}{12} 3 h^2 \frac{dh}{dt} \]

\[ 9 = \frac{\pi}{12} 3 \cdot 6^2 \frac{dh}{dt} \]

\[ 9 = 9 \pi \frac{dh}{dt} \]

\[ \frac{dh}{dt} = \frac{9}{9 \pi} = \frac{1}{\pi} \text{ feet/min} \]