1. You are designing a window consisting of a rectangle with a half-circle on top, as illustrated. The client can only afford 1 meter of window framing material. The framing material runs around the outside of the window and between the rectangular and semicircular regions. What should the diameter of the half-circle be to maximize the area of the window?

(a) Label the diagram with the appropriate variables. Find the function to be optimized.

\[ 1 = h + d + h + d + \frac{\pi d}{2} \]
\[ \text{Constraint: Framing material} \]
\[ 1 = 2h + (2 + \frac{\pi}{2})d \quad \text{(relationship)} \]
\[ h = \frac{1}{2} - (1 + \frac{\pi}{4})d \]

(b) Find the critical points of this function.

\[ A'(d) = \frac{\pi}{4}d + \frac{1}{2} - (2 + \frac{\pi}{2})d \quad \text{(never undefined)} \]
\[ 0 = \frac{\pi}{4}d + \frac{1}{2} - (2 + \frac{\pi}{2})d \quad (A' = 0) \]
\[ \frac{-1}{2} = \frac{\pi}{4}d - (2 + \frac{\pi}{2})d \]
\[ \frac{-1}{2} = d \left( \frac{\pi}{4} - 2 - \frac{\pi}{2} \right) = d \left( -2 - \frac{\pi}{4} \right) \Rightarrow d = \frac{1}{4 + \frac{\pi}{2}} \]

(c) Use the first or second derivative test on the critical points that make sense in the context of this problem.

\[ A''(d) = \frac{\pi}{4} + \frac{1}{2} - 2 - \frac{\pi}{2} = \frac{-3}{2} - \frac{\pi}{4} < 0 \quad \text{for all } d \]

In particular, \( A''(\text{critical pt}) < 0 \) so by the second derivative test, \( d = \frac{1}{4 + \frac{\pi}{2}} \) is the location of a maximum.

(d) Answer the question.

The diameter should be \( \frac{1}{4 + \frac{\pi}{2}} \) meters.
1. You are designing a cylindrical can which has a bottom but no lid. The can must have a volume of 1000 cm$^3$. What should the height and radius of the can be to minimize its surface area?

(a) Label the diagram with the appropriate variables. Find the function to be optimized.

Constraint - volume

$V = 1000 = \pi r^2 h$

$\frac{1000}{\pi r^2} = h$

(relationship between $h$ and $r$)

Function to be optimized

$A = \pi r^2 + 2\pi rh$ (two variables!)

$A(r) = \pi r^2 + 2\pi \left(\frac{1000}{\pi r^2}\right)$

A single variable function

(b) Find the critical points of this function.

$A'(r) = 2\pi r - \frac{2000}{r^2}$

$O = 2\pi r - \frac{2000}{r^2}$ ($A' = 0$)

$2\pi r = \frac{2000}{r^2}$

$2\pi r^3 = 2000$ so $r = \sqrt[3]{\frac{1000}{\pi}} = \frac{10}{\sqrt[3]{\pi}}$ (critical point)

(c) Use the first or second derivative test on the critical points that make sense in the context of this problem.

We don't want a can with radius = 0, so ignore $r = 0$ critical point $A''(r) = 2\pi + \frac{4000}{r^3} > 0$ for all positive $r$

In particular, $A''$ (critical point) > 0 so by the second derivative test, $r = \frac{10}{\sqrt[3]{\pi}}$ is the location of a minimum of $A(r)$

(d) Answer the question.

$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi} \frac{\pi^{-2/3}}{100} = 10 \pi^{-5/3} \text{ cm}$ and $r = \frac{10}{3\sqrt[3]{\pi}} \text{ cm}$
1. You are designing a cylindrical can (with both a top and a bottom) that must have a volume of 1000 cm³. What should the height and radius of the can be to minimize its surface area?

![Diagram of a cylinder with label h and r]

(a) Label the diagram with the appropriate variables. Find the function to be optimized.

Constraint - volume
\[ V = 1000 = h \cdot \pi r^2 \]

\[ \frac{1000}{\pi r^2} = h \]

(relationship between h and r)

Function to be optimized
\[ A = \pi r^2 + \pi r^2 + 2\pi rh \]

(area of top & bottom)

\[ A = 2\pi r^2 + 2\pi rh \]

(two ways!)

\[ A(r) = 2\pi r^2 + \pi \frac{1000}{r^2} \]

\[ A(r) = 2\pi r^2 + \frac{2000}{r} \]

(single variable function)

(b) Find the critical points of this function.

\[ A'(r) = 4\pi r - \frac{2000}{r^2} \]

undefined at \( r = 0 \) (but \( r = 0 \) not in domain of \( A \), so not a critical point)

\[ 0 = 4\pi r - \frac{2000}{r^2} \quad (A' = 0) \]

\[ \frac{2000}{r^2} = 4\pi r \]

\[ 2000 = 4\pi r^3 \quad \text{so} \quad r = \frac{\sqrt[3]{5000}}{\pi} \quad \text{(critical point)} \]

(c) Use the first or second derivative test on the critical points that make sense in the context of this problem.

We don't want a can with radius = 0, so don't bother with \( r = 0 \)

\[ A''(r) = 4\pi + \frac{4000}{r^3} > 0 \quad \text{for all positive radius} \]

in particular, \( A''(\text{critical point}) > 0 \) so by the second derivative test, \( r = \frac{\sqrt[3]{5000}}{\pi} \) is the location of a minimum of \( A(r) \)

(d) Answer the question.

\[ h = \frac{1000}{\pi r^2} = \frac{1000}{\pi} \left( \frac{500}{\pi} \right)^{-\frac{3}{2}} \text{ cm} \]

and \( r = \frac{\sqrt[3]{5000}}{\pi} \text{ cm} \)
1. You are designing a window consisting of a rectangle with a half-circle on top, as illustrated. The client can only afford 1 meter of window framing material which will run along the very outside portion of the window; no framing material is required between the rectangular and semicircular regions. What should the diameter of the half-circle be to maximize the area of the window?

(a) Label the diagram with the appropriate variables. Find the function to be optimized.

\[ A = \frac{1}{2} \pi r^2 + h d \]

Remember. radius is half of diameter

\[ A(d) = \frac{1}{2} \pi \left( \frac{d}{2} \right)^2 + \frac{h d}{2} \quad \text{(two variables!)} \]

\[ A(d) = \frac{\pi}{8} d^2 + \frac{1}{2} d - \left( \frac{1}{2} + \frac{\pi}{4} \right) d^2 \]

(b) Find the critical points of this function.

\[ A'(d) = \frac{\pi}{4} d + \frac{1}{2} - \left( 1 + \frac{\pi}{2} \right) d \quad \text{(never undef)} \]

\[ 0 = \frac{\pi}{4} d + \frac{1}{2} - \left( 1 + \frac{\pi}{2} \right) d \quad \text{(} A' = 0 \text{)} \]

\[ -\frac{1}{2} = \frac{\pi}{4} d - d - \frac{\pi}{2} d = -d - \frac{\pi}{4} d = d \left( -1 - \frac{\pi}{2} \right) \]

\[ d = \frac{1}{2 + \frac{\pi}{2}} \quad \text{so} \quad d = \frac{1}{2 + \frac{\pi}{2}} \quad \text{is the critical point of} \ A(d) \]

(c) Use the first or second derivative test on the critical points that make sense in the context of this problem.

\[ A''(d) = \frac{\pi}{4} - (1 + \frac{\pi}{2}) = -1 - \frac{\pi}{4} < 0 \quad \text{for all} \ d \]

in particular, \[ A''(\text{critical point}) < 0 \]

so by second derivative test, \[ d = \frac{1}{2 + \frac{\pi}{2}} \]

is the location of a maximum of the area function.

(d) Answer the question.

The diameter should be \[ \frac{1}{2 + \frac{\pi}{2}} \] meters.