Section 5.3 The Fundamental Theorem of Calculus (Continued)

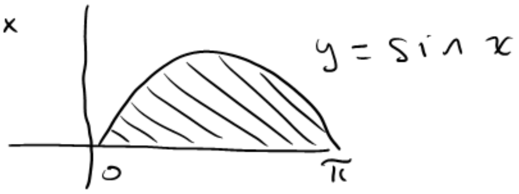
Recall

Suppose $f(x)$ is continuous on $[a,b]$. Then:

F.T.C. I	$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$
F.T.C. II	If $F(x)$ is any antiderivative of $f(x)$, then $\int_a^b f(x)\,dx = F(b)-F(a)$.

F.T.C. II is particularly useful because it gives a formula for the definite integral $\int_a^b f(x)\,dx$

Example Find the area:



Answer will be $\int_0^\pi \sin x\,dx$.

To find this value, FTC II says we need an antiderivative of $\sin x$. This is easy. The antiderivatives are $-\cos x + C$. But FTC says we can use any antiderivative, say $F(x) = -\cos x + 0$

Thus area is $\int_0^\pi \sin x\,dx = -\cos(\pi) - (-\cos(0)) = -(-1)+1 = \boxed{2 \text{ sq. units}}$

Notation Given a function $F(x)$, we define $F(b)-F(a) = F(x)\Big|_a^b$

Thus F.T.C II says $\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b)-F(a)$

Also used $\left[F(x)\right]_a^b = F(b)-F(a)$

Ex $\displaystyle\int_1^e \frac{1}{x}\,dx = \ln|x|\Big|_1^e = \ln|e| - \ln|1| = 1-0 = \boxed{1}$

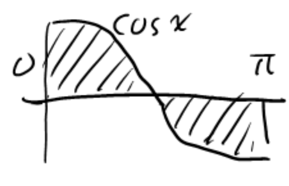
Ex $\displaystyle\int_2^3 x^3\,dx = \frac{x^4}{4}\Big|_2^3 = \frac{3^4}{4} - \frac{2^4}{4} = \frac{81}{4} - \frac{16}{4} = \boxed{\frac{65}{4}}$

Ex $\displaystyle\int_{-1}^1 \frac{1}{1+x^2}\,dx = \tan^{-1}(x)\Big|_{-1}^1 = \tan^{-1}(1) - \tan^{-1}(-1) = \frac{\pi}{4} - \frac{-\pi}{4} = \boxed{\frac{\pi}{2}}$

Ex $\displaystyle\int_0^\pi \cos x\,dx = \sin x\Big|_0^\pi = \sin\pi - \sin 0 = 0-0 = \boxed{0}$

Note: Answer makes sense:

$\int_0^\pi \cos x\,dx = A_{up} - A_{down} = 0$



A few of the exercises require that you use the Fundamental Theorem of Calculus, part I.

$$\boxed{\underline{\text{FTC I}} \qquad \frac{d}{dx}\left[\int_a^x f(t)\, dt \right] = f(x)}$$

$\underline{\text{Ex}}$ $\frac{d}{dx}\left[\int_3^x \cos(t)\, dt \right] = \frac{d}{dx}\left[\sin(t) \Big|_3^x \right] = \frac{d}{dt}\left[\sin(x) - \sin(3) \right]$

$= \cos(x) - 0 = \cos(x)$

But it even works when you can't find the antiderivative, as the following examples show.

<u>Examples</u>

- $$\frac{d}{dx}\left[\int_{-5}^{x} \frac{\ln(t^2+1)\cos(t)}{t^5 + e^t}\, dt \right] = \boxed{\frac{\ln(x^2+1)\cos(x)}{x^5 + e^x}}$$

- $$\frac{d}{dx}\left[\int_x^2 \frac{t\sin(t)}{e^t}\, dt \right] = \frac{d}{dt}\left[-\int_2^x \frac{t\sin(t)}{e^t}\, dt \right]$$
$$= \boxed{-\frac{x\sin(x)}{e^x}}$$

- $$\frac{d}{dx}\left[\int_0^{x^2+x} t\cos t\, dt \right] = ?$$ ← Chain rule problem

$$y = \int_0^{x^2+x} t\cos t\, dt \qquad \begin{cases} y = \int_0^u t\cos t\, dt \\ u = x^2 + x \end{cases}$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{d}{du}\left[\int_0^u t\cos(t)\, dt \right] \frac{d}{dx}\left[x^2 + x \right]$$
$$= u\cos(u)(2x+1)$$
$$= \boxed{(x^2+x)\cos(x^2+x)(2x+1)}$$

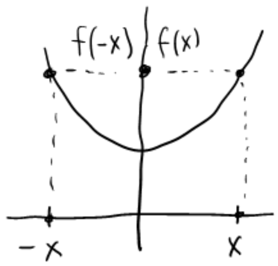
Section 5.4 Working With Integrals

Even and Odd Functions

Here's a topic that may occasionally be useful.

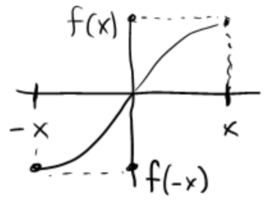
Definitions

$f(x)$ is an <u>even function</u> if $f(-x) = f(x)$ for every x in its domain



even functions are symmetric about the y-axis

$f(x)$ is an <u>odd function</u> if $f(-x) = -f(x)$ for every x in its domain



odd functions are symmetric "about the origin"

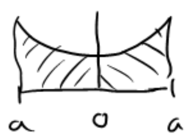
<u>Even Functions</u> $f(x) = x^2$ $\quad f(x) = x^4$ $\quad f(x) = \cos(x)$ $\quad$ etc.

<u>Odd Functions</u> $f(x) = x^3$ $\quad f(x) = x^5$ $\quad f(x) = \sin(x)$ $\quad$ etc

<u>Neither even nor odd</u> $f(x) = x^2 + x$ $\quad f(-x) = (-x)^2 + (-x) = x^2 - x \neq \pm f(x)$

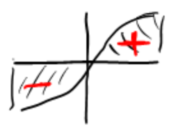
<u>Theorem 5.4</u> Suppose $f(x)$ is continuous on $[-a, a]$.

If $f(x)$ is even, $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$



follows from picture

If $f(x)$ is odd, $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$



follows from picture.– areas cancel

Why would we care? For one thing this can <u>sometimes</u> help us find a definite integral even when we can't find the antiderivative.

<u>Ex</u> $\displaystyle\int_{-5}^{5} x\cos x\,dx = ?$

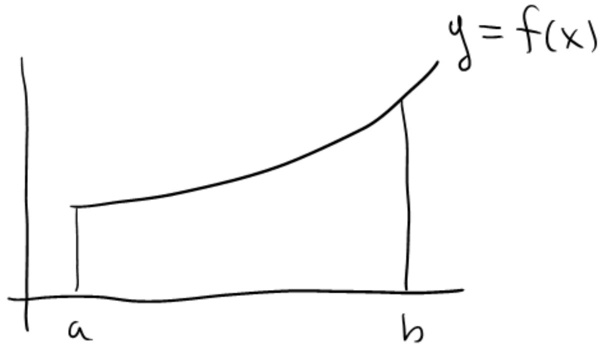
You may be hard pressed to find an antiderivative of $x\cos x$. However, note that $f(x) = x\cos x$ is <u>odd</u>

$f(-x) = (-x)\cos(-x) = -x\cos x = -f(x)$

Then theorem says $\displaystyle\int_{-5}^{5} x\cos x\,dx = \boxed{0}$

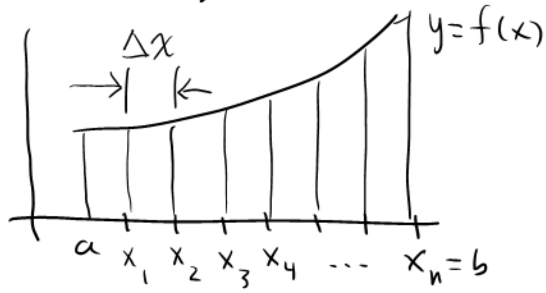
Application The average value of f(x) on [a,b]

Question What is the average value of f(x) on [a,b] ?



Under normal circumstances we would add up all the f(x) values and divide by the number of values.

But this makes no sense because there are ∞ many f(x) values.



To resolve this difficulty take x-values $a < x_1 < x_2 < x_3 \cdots < x_n = b$ as illustrated.

As usual, put $\Delta x = \dfrac{b-a}{n}$

Average value of f(x) on [a,b] is approximately

$$\frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n} = \sum_{k=1}^{n} f(x_k)\frac{1}{n} = \frac{1}{b-a}\sum_{k=1}^{n} f(x_k)\frac{b-a}{n} = \boxed{\frac{1}{b-a}\sum_{k=1}^{n} f(x_k)\Delta x}$$

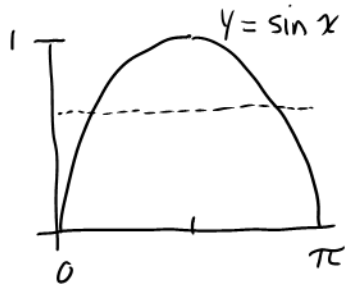
To get the exact ave. value, let the number n of sample points go to ∞:

$$\left(\text{Ave. Value of } f(x) \text{ on } [a,b]\right) = \lim_{n\to\infty}\frac{1}{b-a}\sum_{k=1}^{n} f(x_k)\Delta x$$

$$= \frac{1}{b-a}\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Conclusion Average value of f(x) on [a,b] is $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$

Ex What's the average value of $\sin x$ on $[0,\pi]$?



Guess: About 0.6 ?

Exact answer: $\dfrac{1}{\pi - 0}\displaystyle\int_0^\pi \sin x\,dx$

$$= -\frac{1}{\pi}\cos(x)\Big|_0^\pi = -\frac{1}{\pi}\cos\pi - \left(-\frac{1}{\pi}\cos 0\right) = \frac{1}{\pi}(1+1) = \boxed{\frac{2}{\pi}} \approx 0.636619$$

You can ignore Theorem 5.5 – Mean Value Theorem for integrals