Section 4.9 Antiderivatives

Finding the derivative of a function has been a major theme for us. Given a function $f(x)$, find its derivative $f'(x)$. That process is called <u>differentiation</u>.

$$f(x) \underset{\text{antidifferentiation}}{\overset{\text{differentiation}}{\rightleftarrows}} f'(x)$$

Today we begin to look at the reverse process: Given $f'(x)$, can we find $f(x)$. This process is called <u>antidifferentiation</u>.

<u>Definition</u> A function $F(x)$ is an <u>antiderivative</u> of $f(x)$ if $F'(x) = f(x)$.

antiderivative of $f(x)$ $\rightarrow$ $$F(x) \underset{\text{antidifferentiate}}{\overset{\text{differentiate}}{\rightleftarrows}} f(x)$$ $\leftarrow$ Derivative of $F(x)$

<u>Ex</u> What is the antiderivative of $f(x) = 2x$?

$$? \xrightarrow{\text{differentiate}} 2x$$

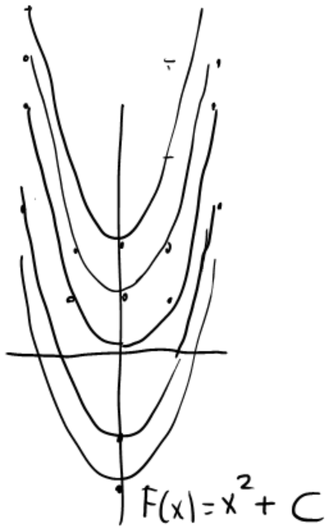
Antiderivative: $\boxed{F(x) = x^2}$

Also: $F(x) = x^2 + 2$

Also: $F(x) = x^2 + C$ where C is <u>any</u> constant.

Thus $f(x) = 2x$ has infinitely many antiderivatives!

some are sketched here



Ex. Find an antiderivative of $f(x) = x^2 + e^{-x} - 3\sin x$

$$\left\{ \tfrac{1}{3}x^3 - e^{-x} + 3\cos x \right\} \xrightarrow{\text{differentiate}} x^2 + e^{-x} - 3\sin x$$

Antiderivative is $F(x) = \frac{1}{3}x^3 - e^{-x} + 3\cos x$

In general, $F(x) = \frac{1}{3}x^3 - e^{-x} + 3\cos x \underline{+ C}$

<u>Definition</u> If $f(x)$ has an antiderivative $F(x)$ (i.e., if $F'(x) = f(x)$) then $F(x) + C$ is called the <u>most general antiderivative of $f(x)$</u> or the <u>indefinite integral of $f(x)$</u>

<u>Note</u> $F(x) + C$ stands for infinitely many functions, one for each possible value of C. Thus $F(x) + C$ is the set of <u>all</u> antiderivatives of $f(x)$.

<u>Notation</u> The indefinite integral of $f(x)$ is denoted as

$$\int f(x)\,dx = F(x) + C.$$

Thus $\int f(x)\,dx$ stands for all antiderivatives of $f(x)$.

<u>Ex</u> $\int x^3\,dx = \frac{1}{4}x^4 + C$

<u>Ex</u> $\int (x^2 + e^{-x} - 3\sin x)\,dx = \frac{1}{3}x^3 - e^{-x} + 3\cos x + C$

<u>Ex</u> $\int \frac{2x}{1+x^2}\,dx = \ln(1+x^2) + C$

<u>Ex</u> $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$

<u>Ex</u> $\int \frac{1}{x}\,dx =$

You want to say this is $\ln x$, but that's not quite right, because $\ln x$ is defined only for <u>positive</u> x, so its <u>derivative</u> $\frac{1}{x}$ is interpreted to have domain $(0,\infty)$. However, here we might think of $\frac{1}{x}$ with domain $(-\infty, 0) \cup (0,\infty)$.

On page 204, text shows $\frac{d}{dx}\left[\ln|x|\right] = \frac{1}{x}$

Thus $\boxed{\int \frac{1}{x}\,dx = \ln|x| + C}$

Never Forget:

$$\boxed{\int f(x)\,dx = F(x)+C \iff \frac{d}{dx}\Big[F(x)+C\Big] = f(x)}$$

Any derivative formula, run in reverse, becomes an antiderivative formula. Here are the main ones:

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \quad \leftarrow \text{Provided } n \neq -1$$

$$\int x^{-1}\,dx = \ln|x| + C \quad \leftarrow \text{Here's what you do when } n = 1.$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C$$

$$\int \csc x \cot x\,dx = -\csc x + C$$

$$\int e^x\,dx = e^x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$$

$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}|x| + C$$

$$\int b^x\,dx = \frac{1}{\ln(b)}b^x + C$$

$$\int k f(x)\,dx = k\int f(x)\,dx$$

$$\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx.$$

Summary:

We now have two opposite operations on functions:

$$\begin{tikzcd} F(x)+C \arrow[r, bend left, "\frac{d}{dx}"] & f(x) \arrow[l, bend left, "\int dx"] \end{tikzcd}$$

More about all this next time!