Section 4.7 Indeterminate Forms and L'Hôpital's Rule (continued)

Recall: L'Hôpital's Rule: If $\lim_{x\to a} \frac{f(x)}{g(x)}$ has indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

Ex $\lim_{x\to 0} \frac{\sin(x) - x}{x^3}$ form $\frac{0}{0}$

$= \lim_{x\to 0} \frac{\cos(x) - 1}{3x^2}$ form $\frac{0}{0}$

$= \lim_{x\to 0} \frac{-\sin(x)}{6x} = -\frac{1}{6} \lim_{x\to 0} \frac{\sin x}{x} = -\frac{1}{6}\cdot 1 = \boxed{-\frac{1}{6}}$

Todays Goal Adapt L'Hôpital's rule to limits of indeterminate forms $\infty - \infty$, $0\cdot\infty$, 0^0, 1^∞ and ∞^0

The Indeterminate form $\infty - \infty$

Ex $\lim_{x\to 0} \left(\frac{1}{e^x - 1} - \frac{1}{x}\right)$ form $\infty - \infty$

$= \lim_{x\to 0} \frac{x - (e^x - 1)}{(e^x - 1)x}$ form $\frac{0}{0}$

$= \lim_{x\to 0} \frac{1 - e^x}{e^x x + (e^x - 1)\cdot 1}$ form $\frac{0}{0}$

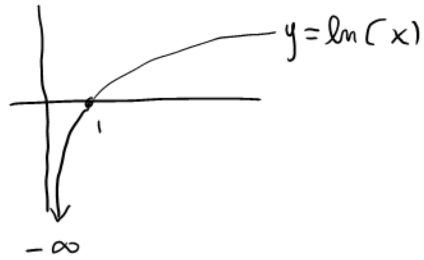
$= \lim_{x\to 0} \frac{-e^x}{e^x x + e^x(1) + e^x} = \lim_{x\to 0} \frac{-e^x}{xe^x + 2e^x} = \frac{-1}{0+2} = \boxed{-\frac{1}{2}}$

Ex $\lim_{x\to\infty} \ln(3x^2 + 1) - 2\ln(x)$

$= \lim_{x\to\infty} \ln(3x^2 + 1) - \ln(x^2) = \lim_{x\to\infty} \ln\left(\frac{3x^2 + 1}{x^2}\right) = \ln\left(\lim_{x\to\infty} \frac{3x^2+1}{x^2}\right) = \boxed{\ln(3)}$

Indeterminate Forms $0 \cdot \infty$

Recall:



Ex $\lim_{x \to 0} x \ln(x)$ form $0 \cdot \infty$

$= \lim_{x \to 0} \frac{\ln(x)}{\frac{1}{x}}$ form $\frac{\infty}{\infty}$

$= \lim_{x \to 0} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x \to 0} (-x) = \boxed{0}$

Ex $\lim_{x \to \frac{\pi}{2}^-} (2x - \pi)\sec(x)$ form $0 \cdot \infty$

$= \lim_{x \to \frac{\pi}{2}^-} \frac{2x - \pi}{\frac{1}{\sec(x)}}$ form $\frac{0}{0}$

$= \lim_{x \to \frac{\pi}{2}^-} \frac{2x - \pi}{\cos(x)} = \lim_{x \to \frac{\pi}{2}^-} \frac{2}{-\sin(x)} = \frac{2}{-1} = \boxed{-2}$

Note could also convert to $\lim_{x \to \frac{\pi}{2}^-} \frac{\sec x}{\frac{1}{2x - \pi}}$ (form $\frac{\infty}{\infty}$) but the derivatives would be more complex.

Ex Find Horizontal asymptotes of $f(x) = xe^x$

$\lim_{x \to \infty} xe^x = \infty$

$\lim_{x \to -\infty} xe^x = \lim_{x \to -\infty} \frac{x}{\frac{1}{e^x}} = \lim_{x \to -\infty} \frac{x}{e^{-x}} = \lim_{x \to -\infty} \frac{1}{-e^{-x}} = 0$

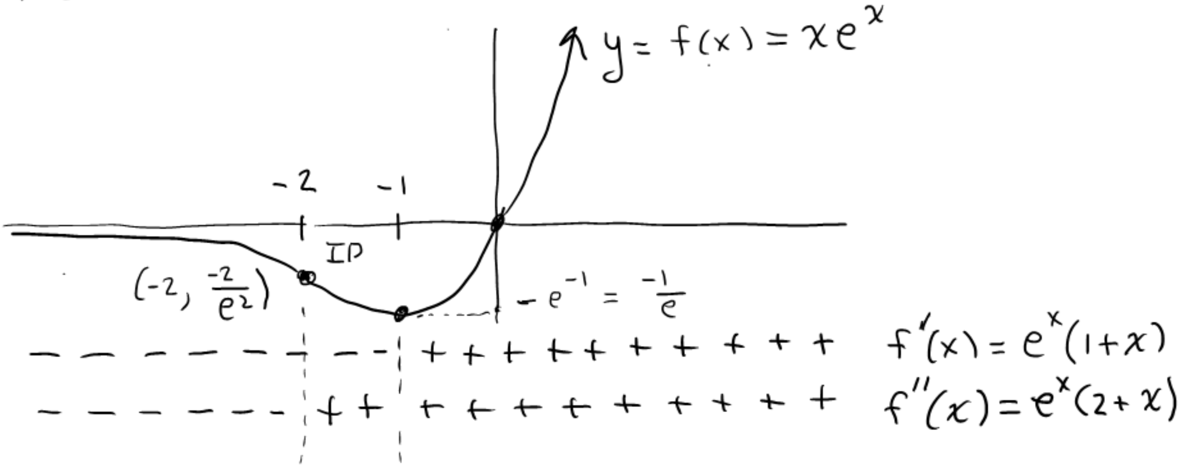
form $\infty \cdot 0$ form $\frac{\infty}{\infty}$ bottom $\to \infty$

Thus $\lim_{x \to -\infty} x e^x = 0$. $\boxed{\text{Line } y = 0 \text{ is HA}}$

Let's round out the discussion by using Ch. 4 techniques to carefully sketch graph of $y = f(x) = xe^x$.

$f'(x) = e^x + xe^x = e^x(1 + x)$

$f''(x) = e^x(1 + x) + e^x(1) = 2e^x + xe^x = e^x(2 + x)$



Indeterminate Forms ∞^0, 0^0, 1^∞

Each of these can be handled with the following trick:

Suppose $\lim_{x\to a} f(x)^{g(x)}$ has form ∞^0, 0^0, or 1^∞. Then:

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{\ln\left(f(x)^{g(x)}\right)} = \lim_{x\to a} e^{g(x)\ln(f(x))}$$

exponent has form $0\cdot\infty$, or $\infty\cdot 0$ and we know how to handle that

Ex $\lim_{x\to\infty} \sqrt[x]{x} = \lim_{x\to\infty} x^{\frac{1}{x}}$ form ∞^0

$$= \lim_{x\to\infty} e^{\ln\left(x^{\frac{1}{x}}\right)} = \lim_{x\to\infty} e^{\frac{1}{x}\ln(x)} = e^{\lim_{x\to\infty}\frac{\ln(x)}{x}}$$

form $\frac{\infty}{\infty}$

$$= e^{\lim_{x\to\infty}\frac{\frac{1}{x}}{1}} = e^{\lim_{x\to\infty}\frac{1}{x}} = e^0 = \boxed{1}$$

Ex $\lim_{x\to 0}(1+x)^{\frac{1}{x}}$ form 1^∞

$$= \lim_{x\to 0} e^{\ln(1+x)^{\frac{1}{x}}} = \lim_{x\to 0} e^{\frac{1}{x}\ln(1+x)} = \lim_{x\to 0} e^{\frac{\ln(1+x)}{x}}$$

$$= \lim_{x\to 0} e^{\frac{\frac{1}{1+x}}{1}} = \lim_{x\to 0} e^{\frac{1}{1+x}} = e^{\frac{1}{1+0}} = e^1 = \boxed{e}$$

[This answer should not be surprising. Earlier we defined e to be the number $\lim_{x\to 0}(1+x)^{1/x}$. Here we worked out the limit and got e. No big surprise.]

Ex $\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\ln x^x} = \lim_{x\to 0^+} e^{x\ln x} = \lim_{x\to 0^+} e^{\frac{\ln x}{\frac{1}{x}}}$ form $\frac{\infty}{\infty}$

form 0^0

$$= \lim_{x\to 0^+} e^{\frac{1}{x}/-\frac{1}{x^2}} = \lim_{x\to 0^+} e^{-x} = e^{-0} = \boxed{1}$$