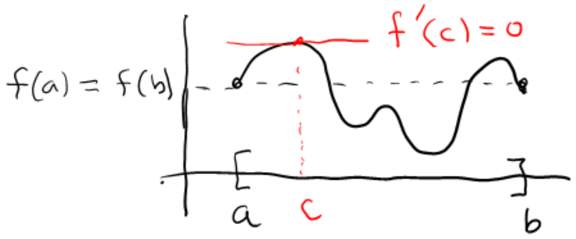
Section 4.6 The Mean Value Theorem

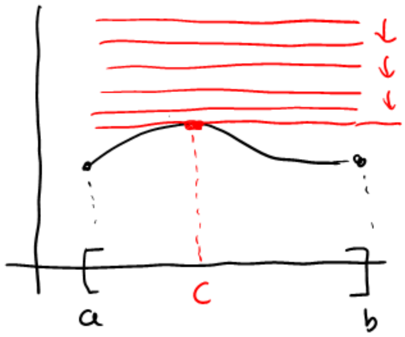
The Mean Value theorem is a significant theoretical result. Although you will not use it often, it lays the foundation for Chapter 5, and is used in proofs of many fundamental results. We first address a preliminary result called Rolle's Theorem

Rolles Theorem Suppose a function $f(x)$ is continuous on an interval $[a,b]$ and differentiable on the interval (a,b). (That is, $f'(x)$ exists and is defined at every number $x \in (a,b)$.) If $f(a)=f(b)$ then there exists at least one number $c \in (a,b)$ for whic $f'(c)=0$.



Rolles Theorem guarantees there is a number with $f'(c)=0$, as illustrated. Note that there may be several such c, as is the case here.

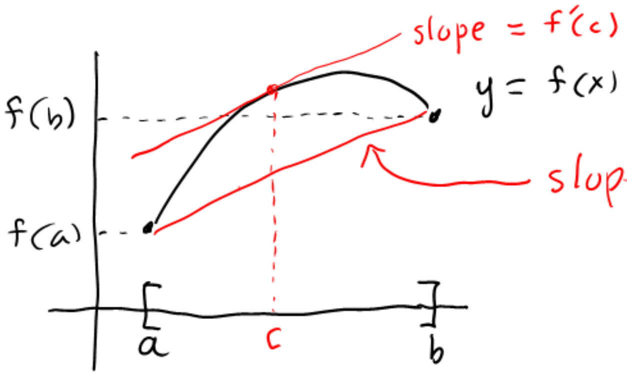
The text gives a careful proof of Rolle's theorem, which you should read. But notice the Theorem is very intuitive. Move a horizontal line down (or up) until it hits a point on the graph of $y=f(x)$. The point of first contact has an x-coordinate c for which $f'(c)=0$, as illustrated here



From Rolles Theorem we get: Theorem 4.9

The Mean Value Theorem Suppose $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b) then there is a $c \in (a,b)$ for which

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

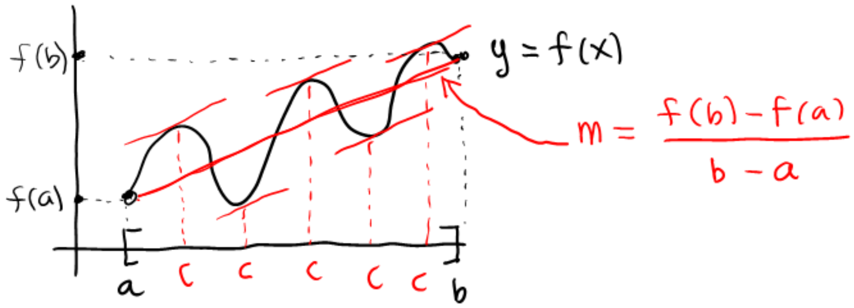


slope = $\dfrac{f(b)-f(a)}{b-a}$

MVT says there is at least one c in (a,b) with $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ i.e. the two slopes are equal.

Read the proof of the MVT in the text, but also note and appreciate that the theorem is very intuitive and common-senseical.

Note that there can be several c in (a,b) with
$f'(c) = \dfrac{f(b)-f(a)}{b-a}$.
MVT says there is at least one such c.



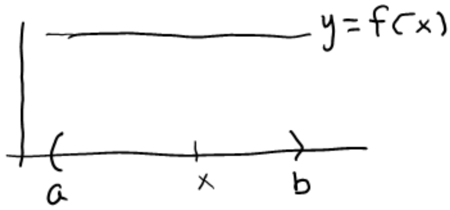
Example (A way of thinking about the MVT.)

Suppose you drive 30 miles in 20 minutes. ($\frac{1}{3}$ hour). Did you break the speed limit? Your intuition say YES because your average velocity is $\dfrac{30 \text{ mi}}{\frac{1}{3} \text{ hour}} = 90$ mph. The mean value confirms this.
Say your position at time t is $s(t)$

MVT says at some time $t = c$, $\underbrace{s'(c)}_{\text{instantaneous velocity at time } c} = \underbrace{\dfrac{s(\frac{1}{3}) - s(0)}{\frac{1}{3} - 0}}_{\text{ave. vel.}} = \dfrac{30}{\frac{1}{3}} = 90$ mph

MVT simply says that at some instant $t = c$, your instantaneous velocity equals your average velocity

Mathematical Consequences of MVT



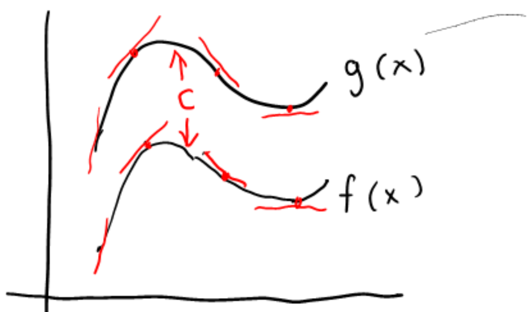
Theorem 4.10 Suppose $f'(x) = 0$ on an interval (a,b). Then $f(x) = C$ on (a,b), where C is a constant.

Proof Take $x \in (a,b)$. By MVT, there exists a $c \in [a, x]$ with $0 = f'(c) = \dfrac{f(a) - f(x)}{a - x}$

$\Rightarrow 0(a-x) = f(a) - f(x)$, i.e. $0 = f(a) - f(x)$. Then $f(x) = -f(a) = C$.

Theorem 4.11 Suppose $f'(x) = g'(x)$ on (a,b). Then $f(x) = g(x) + C$ for some constant c.

Proof If $f'(x) = g'(x)$ on (a,b), then $f'(x) - g'(x) = 0$ on (a,b). That is, $(f+g)'(x) = 0$ on (a,b). By corollary 1, $(f+g)x = C$, i.e. $f(x) + g(x) = C$
Thus $f(x) = g(x) + (-C)$



Corollary says that if $f'(x) = g'(x)$, i.e. slopes of $f(x)$ and $g(x)$ agree then $f(x) = g(x) + C$