Chapter 4 Applications of Derivatives

Now that we know how to find derivatives, and what they mean, we turn to some applications. The main topic of chapter 4 involves finding maximum and minimum values of functions.

Examples Profit $= P(x)$ ← Goal find x that maximizes this.

Cost $= C(x)$ ← Goal find x that minimizes this

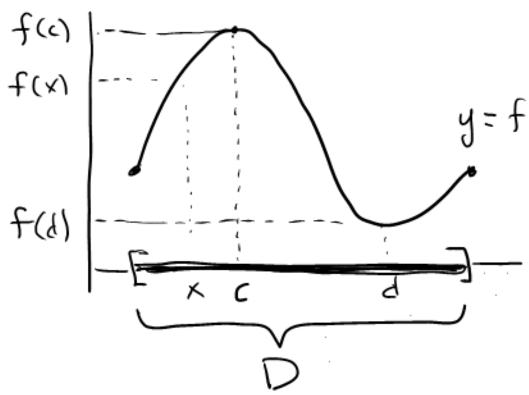
In general, if $f(x)$ is a good or desirable quantity, we want to find an x that maximizes $f(x)$ (i.e. makes it as large as possible). If $f(x)$ is a bad or undisrable quantity, we seek an x that minimizes it (i.e. makes it as small as possible). Thus our goal is to find an optimal outcome. Derivatives (as we will see) provide a means of doing this. Section 4.1 sets up some main ideas and definitions.

Section 4.1 Maxima and Minima

Definitions Suppose $f(x)$ is defined on a domain D (usually an interval).

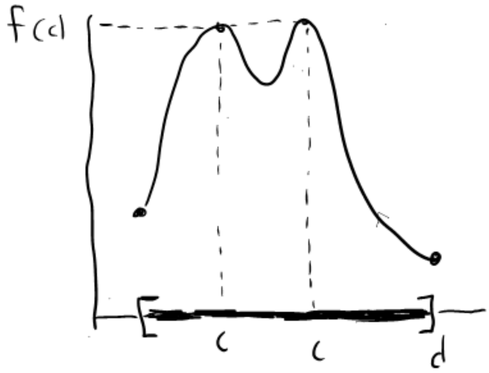
- $f(x)$ has an absolute maximum at c in D if $f(x) \le f(c)$ for all x in D.
- $f(x)$ has an absolute minimum at c in D if $f(x) \ge f(c)$ for all x in D.

Examples



$f(x)$ has absolute maximum at c because $f(x) \le f(c)$ for all x.

$f(x)$ has absolute minimum at d because $f(x) \ge f(d)$ for all x.



This function has an absolute maximum of $f(c)$, which is attained at two separate values of c.

The absolute minimum occurs at d, an endpoint of D.

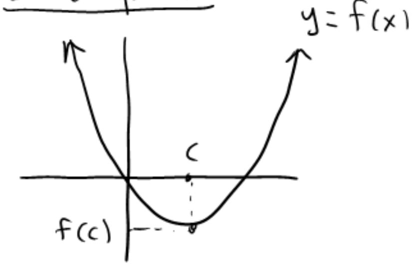
Terminology:

If $f(x)$ has an abs. max at c, $f(c)$ is an absolute maximum of $f(x)$ at c

If $f(x)$ has an abs. min. at d, $f(d)$ is an absolute minimum of $f(x)$ at d

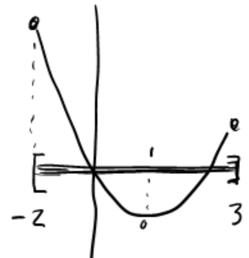
$f(c)$ and $f(d)$ are absolute extrema

Examples



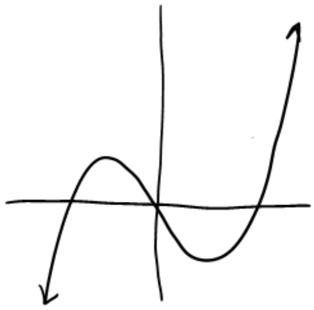
$D = (-\infty, \infty) = \mathbb{R}$

Absolute minimum at c.
No absolute maximum



$D = [-2, 3]$

Absolute maximum at -2
Absolute minimum at 1

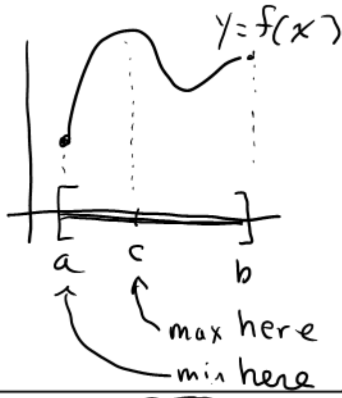


$D = (-\infty, \infty) = \mathbb{R}$

No absolute max.
No absolute min.

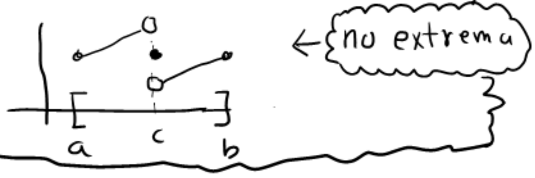
So in some circumstances absolute extrema don't exist. But there is one situation in which it is guaranteed:

Theorem 4.1 (Extreme Value Theorem)
If $f(x)$ is continuous on a closed interval $[a, b]$, then it has both an absolute max. and an absolute min. on $[a, b]$

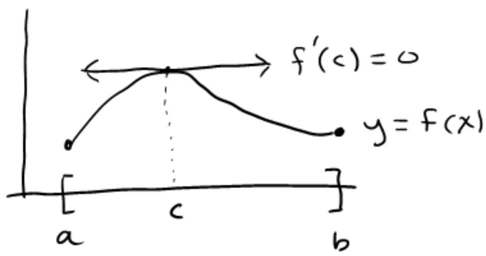


Intuitively this seems plausible, almost obvious, but the proof is subtle and is left to more advanced courses.

Note that the condition of continuity is essential, as this picture suggests:



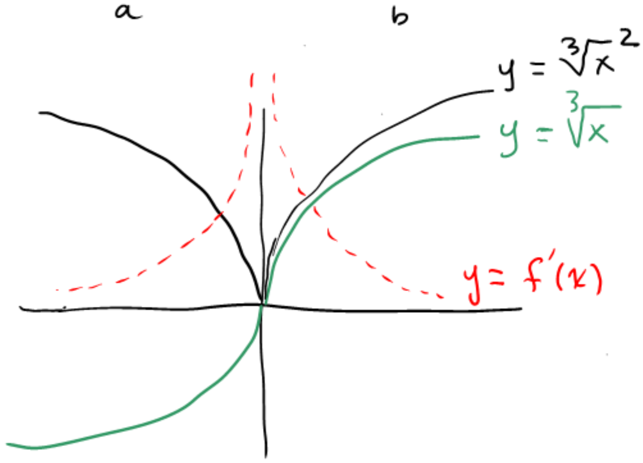
Our goal is to locate extreme values. Where do they occur? Our pictures suggest that extreme values can happen at endpoints and values of c where $f'(c) = 0$.



Where else? An example is instructive

Consider $f(x) = \sqrt[3]{x}^2 = x^{2/3}$

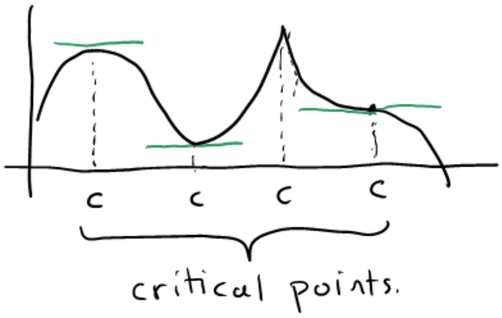
$$f'(x) = \frac{2}{3} x^{2/3 - 1} = \frac{2}{3} x^{-\frac{1}{3}} = \frac{2}{3x^{1/3}} = \frac{2}{3\sqrt[3]{x}}$$



Note that $f(x) = \sqrt[3]{x}^2$ has an absolute minimum at $c = 0$ and $f'(0)$ is not defined.

Conclusion Extreme values can occur at endpoints or values c for which $f'(c) = 0$ or $f'(c)$ is undefined

This leads to a definition



Definition A number c in the domain of $f(x)$ is called a critical point if $f'(c)=0$ or $f'(c)$ is undefined.

Note critical points are candidates for extreme values.

This leads to our main result of the day:

How to find the extreme values of $f(x)$ on a closed interval $[a,b]$.

1. Find all critical points c in (a,b).
2. Evaluate each $f(c)$ and $f(a)$ and $f(b)$
3. Select the largest value from Step 2. It's the abs. max
   Select the smallest value from Step 2 It's the abs. min.

Example Find absolute extrema of $f(x)=\sqrt[3]{x}^{\,2}-\frac{4}{3}x$ on $[0, 27]$

1. Find critical pts. of $f(x)$

To do this, examine $f'(x)=\dfrac{2}{3\sqrt[3]{x}}-\dfrac{4}{3}$

Note $f'(0)$ is undefined, so 0 is a critical point.

The other critical points will be those $x=c$ for which $f'(c)=0$

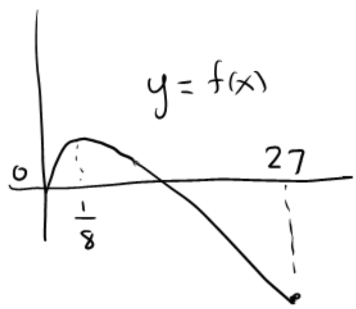
To find them, set $f'(x)=0$ and solve for x.

$$f'(x)=0$$
$$\frac{2}{3\sqrt[3]{x}}-\frac{4}{3}=0$$
$$\frac{2}{3}\left(\frac{1}{\sqrt[3]{x}}-2\right)=0$$
$$\frac{1}{\sqrt[3]{x}}-2=0$$
$$\frac{1}{\sqrt[3]{x}}=2$$
$$\sqrt[3]{x}=\frac{1}{2}$$
$$x=\left(\frac{1}{2}\right)^3=\frac{1}{8}$$

Thus the critical points are $c=0$ and $c=\frac{1}{8}$.

Note $c=0$ just happens to also be an endpoint of $[0,27]$

2.
$$\left.\begin{array}{l} f(0)=\sqrt[3]{0}^{\,2}-\frac{4}{3}\cdot 0=0 \\ f\left(\frac{1}{8}\right)=\sqrt[3]{\frac{1}{8}}^{\,2}-\frac{4}{3}\cdot\frac{1}{8}=\frac{1}{4}-\frac{1}{6}=\frac{3}{12}-\frac{2}{12}=\frac{1}{12} \end{array}\right\}\text{critical pts.}$$
$$\left.\begin{array}{l} f(0)=0 \\ f(27)=\sqrt[3]{27}^{\,2}-\frac{4}{3}\cdot 27=9-36=-27 \end{array}\right\}\text{end pts.}$$



3.
$$\boxed{\begin{array}{l} f\left(\frac{1}{8}\right)=\frac{1}{12} \text{ is abs. max.} \\ f(27)=-27 \text{ is abs. min.} \end{array}}$$

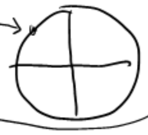
Example Find absolute extrema of $f(x) = x(\sin x - \cos x) + \cos x + \sin x$ on $[\frac{\pi}{2}, \pi]$

① $f'(x) = (1)(\sin x - \cos x) + x(\cos x + \sin x) - \sin x + \cos x$

$f'(x) = x(\cos x + \sin x) = 0$ ← $f'(x)$ is defined for all x so only critical points are those c for which $f'(c) = 0$. To find them we set $f'(x) = 0$ and solve for x.

$x = 0$ $\qquad$ $\cos x + \sin x = 0$

$x = \frac{3\pi}{4}$

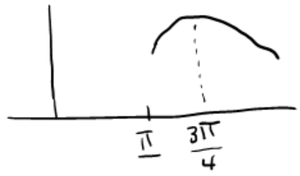


Thus critical points are 0 and $\frac{3\pi}{4}$, but note 0 is not in interval.

② $f\left(\frac{3\pi}{4}\right) = \frac{3\pi}{4}\left(\sin\frac{3\pi}{4} - \cos\frac{3\pi}{4}\right) + \cos\left(\frac{3\pi}{4}\right) + \sin\left(\frac{3\pi}{4}\right) = \frac{3\pi}{4}\sqrt{2} \approx 3.33$ ← ABS MAX

$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\left(\sin\frac{\pi}{2} - \cos\frac{\pi}{2}\right) + \cos\frac{\pi}{2} + \sin\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + 1 \approx 2.57$

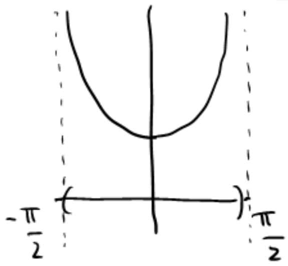
$f(\pi) = \pi(\sin\pi - \cos\pi) + \cos\pi + \sin\pi = \pi - 1 \approx 2.14$ ← ABS MIN



③ Abs max is $f\left(\frac{3\pi}{4}\right) = \frac{3\pi\sqrt{2}}{4}$

Abs min is $f(\pi) = \pi - 1$

If $f(x)$ is defined on a non-closed interval, (e.g. an open interval), then the above procedure breaks down. For example, consider $f(x) = \sec(x)$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$. It has no absolute maximum.

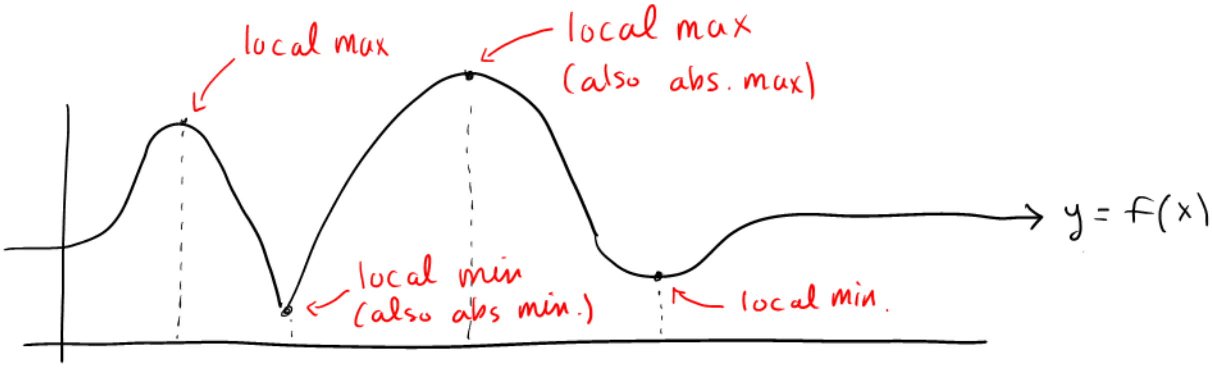


In fact, finding extrema on open intervals is sometimes tricky. As a step towards resolving this issue, we introduce the idea of <u>local extrema</u>

Definitions

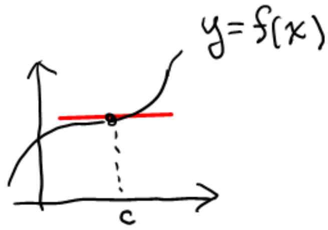
- $f(x)$ has a <u>local maximum</u> at c if $f(x) \leq f(c)$ for all x <u>near c</u>
- $f(x)$ has a <u>local minimum</u> at c if $f(x) \geq f(c)$ for all x <u>near c</u>

Example



Text proves that any local extremum that is not an endpoint must occur at a <u>critical point</u>. (See Theorem 4.2)

But don't forget the critical points are just the <u>candidates</u> for locations of local extrema. You can have a critical point at which there is neither a local max nor a local min, as illustrated →



In section 4.2 we'll explore the <u>First derivative test</u>. It tells which critical points yield local max, min or neither.