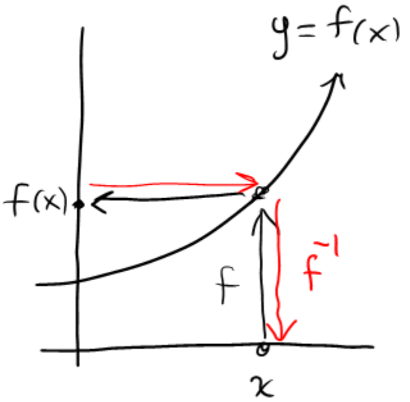
Review of Inverse Functions

Basic Idea

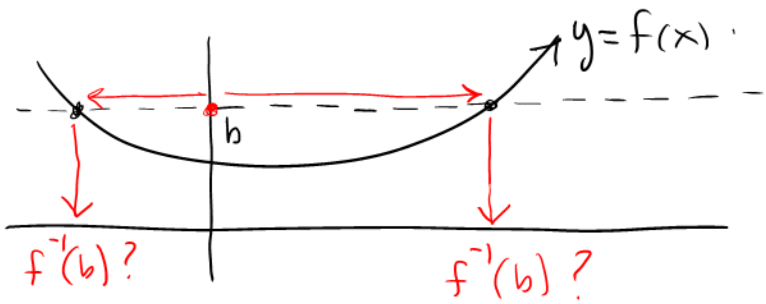
Given a function $f(x)$, its *inverse* is a function $f^{-1}(x)$ that "undoes" the action of $f(x)$ in the sense

$$f^{-1}(f(x)) = x$$



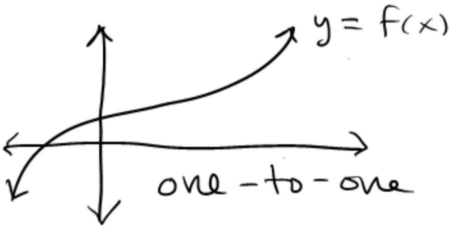
The inverse $f^{-1}(x)$ is $f(x)$ in reverse. It sends $f(x)$ right back to x.

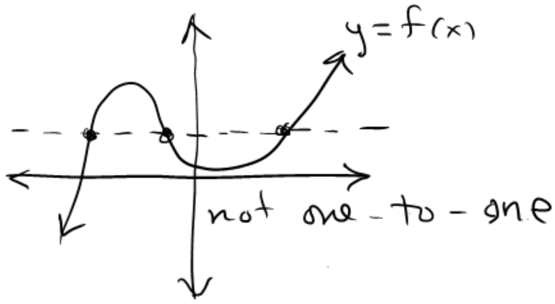
But for this to work, no horizontal line can cross the graph of $y = f(x)$ more than once



Definitions

- A function is one-to-one if no horizontal line crosses its graph more than once.





- If $f(x)$ is one-to-one, then it has an inverse $f^{-1}(x)$ satisfying:

$$f^{-1}(f(x)) = x$$

$$f(f^{-1}(x)) = x.$$

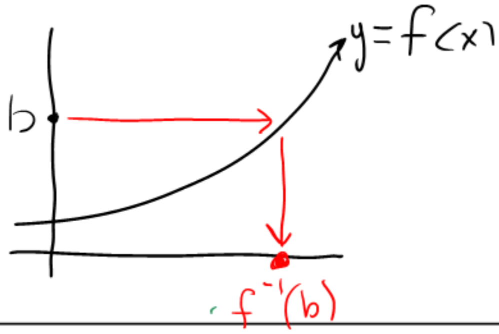
$$[\text{Domain of } f^{-1}(x)] = [\text{Range of } f(x)]$$

$$[\text{Range of } f^{-1}(x)] = [\text{Domain of } f(x)]$$

Always keep the following basic interpretation in mind:

$$f^{-1}(b) = \left(\begin{array}{c}\text{what you need to plug} \\ \text{into } f(x) \text{ to get } b\end{array}\right)$$

$$f^{-1}(x) = \left(\begin{array}{c}\text{what you need to plug} \\ \text{into } f(x) \text{ to get } x\end{array}\right)$$



Example

$$f(x) = x + 2^x$$

$$f^{-1}(11) = \begin{pmatrix}\text{number } y \text{ for} \\ \text{which } f(y)=11\end{pmatrix} = \begin{pmatrix}\text{number } y \text{ for} \\ \text{which } y+2^y=11\end{pmatrix} = 3$$

$$f^{-1}(6) = \begin{pmatrix}\text{number } y \text{ for} \\ \text{which } f(y)=6\end{pmatrix} = 2$$

$$f^{-1}(7) = [\text{not so easy!}]$$

One reason inverses are useful:

Problem Solve equation: $f(x) = y$ for x

Solution Apply inverse: $f^{-1}(f(x)) = f^{-1}(y)$

$$\boxed{x = f^{-1}(y)}$$

So if we solve $y = f(x)$ for x we get $x = f^{-1}(x)$

How to compute the inverse of a function $f(x)$:

1. Write $y = f(x)$
2. Swap variables: $x = f(y)$
3. Solve for y. Get $y = f^{-1}(x)$.

Example $f(x) = \dfrac{2x-1}{x-3}$ Find its inverse.

1. $y = \dfrac{2x-1}{x-3}$

2. $x = \dfrac{2y-1}{y-3}$

3. $x(y-3) = 2y-1$

$xy - 3x = 2y - 1$

$xy - 2y = 3x - 1$

$y(x-2) = 3x - 1$

$y = \dfrac{3x-1}{x-2} \longrightarrow \boxed{f^{-1}(x) = \dfrac{3x-1}{x-2}}$

Check: Do $f^{-1}(f(x))$. should get x back

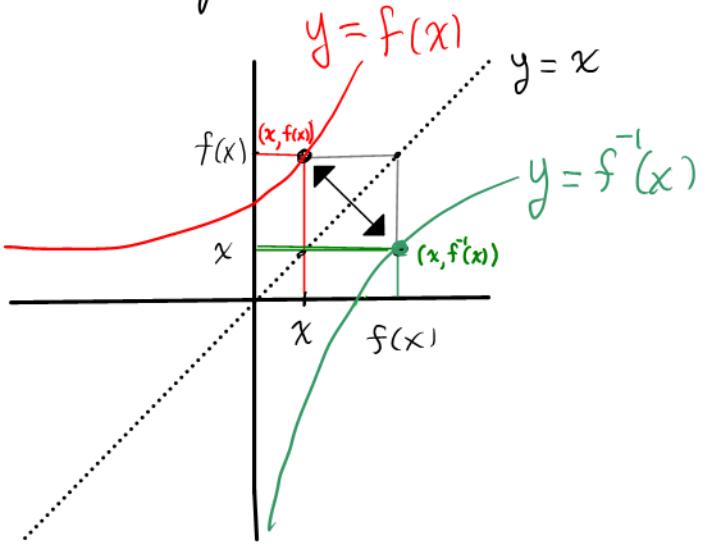
$$f^{-1}(f(x)) = \frac{3f(x)-1}{f(x)-2} = \frac{3\frac{2x-1}{x-3}-1}{\frac{2x-1}{x-3}-2} = \frac{\frac{3(2x-1)-(x-3)}{x-3}}{\frac{2x-1-2(x-3)}{x-3}} = \frac{\frac{5x}{x-3}}{\frac{5}{x-3}} =$$

$$= \frac{5x}{x-3}\,\frac{x-3}{5} = x$$

YES! Got $f^{-1}(f(x)) = x$. Checks back!

Fact:

Graph of $f^{-1}(x)$ is graph of $f(x)$ reflected across line $y = x$



Exponential Functions

Laws of Exponents

- $a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ times}}$
- $a^{-n} = \dfrac{1}{a^n}$
- $a^{\frac{m}{n}} = \sqrt[n]{a}^{\,m} = \sqrt[n]{a^m}$
- $a^m a^n = a^{m+n}$
- $(ab)^n = a^n b^n$
- $a^1 = a$
- $\dfrac{a^m}{a^n} = a^{m-n}$
- $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$
- $a^0 = 1 \quad (\text{if } a \neq 0)$

Example: $16^{-1.5} = \dfrac{1}{16^{1.5}} = \dfrac{1}{16^{3/2}} = \dfrac{1}{\sqrt[2]{16}^{\,3}} = \dfrac{1}{4^3} = \boxed{\dfrac{1}{64}}$

Definition

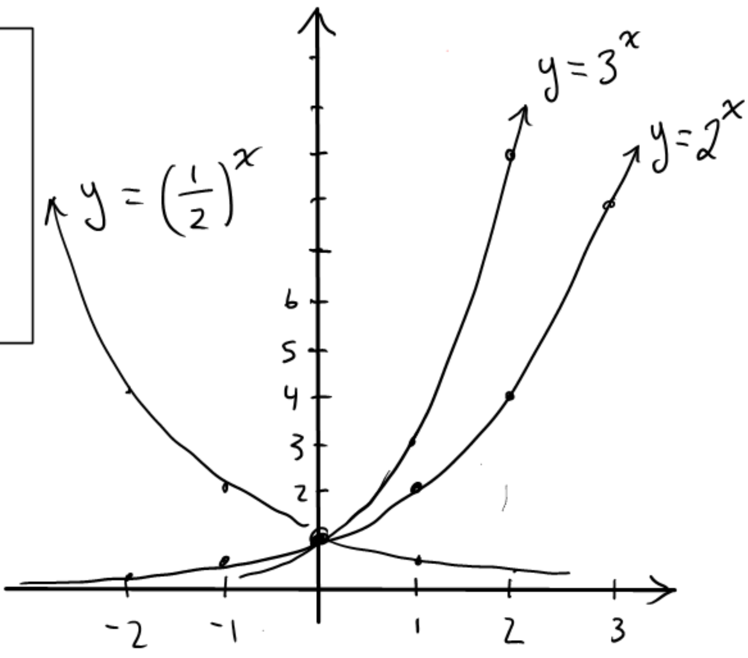
A function of form $f(x) = a^x$ with a positive, $a \neq 0$, is called an exponential function

Examples

$f(x) = 2^x$

$f(x) = 3^x$

$f(x) = \left(\frac{1}{2}\right)^x$

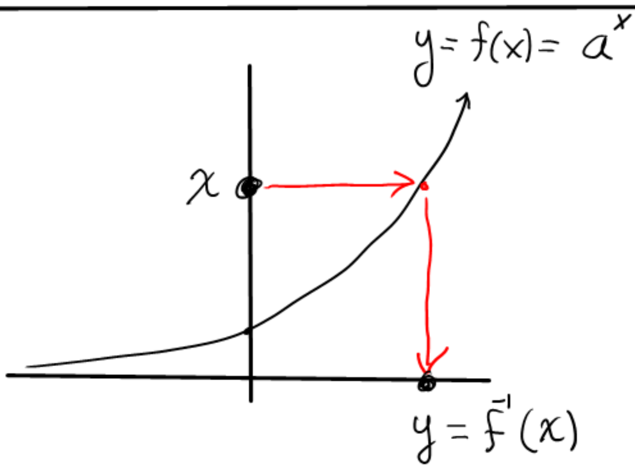


Logarithms

Any exponential function $f(x) = a^x$ is one-to-one and thus has an inverse

$$f^{-1}(x) = \begin{pmatrix} \text{number } y \text{ for} \\ \text{which } f(y) = x \end{pmatrix}$$

$$= \begin{pmatrix} \text{number } y \text{ for} \\ \text{which } a^y = x \end{pmatrix}$$



It makes sense to call this inverse $a^{\square}$ instead of f^{-1}

$$a^{\square}(x) = \begin{pmatrix} \text{number } y \text{ for} \\ \text{which } a^y = x \end{pmatrix} = \left\{ \begin{matrix} \text{what goes in box so} \\ a \text{ to that power is } x \end{matrix} \right\}$$

Examples

Inverse of $f(x) = 2^x$ is the function $2^{\square}(x) = \begin{pmatrix} \text{number } y \text{ for} \\ \text{which } 2^y = x \end{pmatrix}$

$2^{\square}(8) = 3$ because $2^3 = 8$

$2^{\square}(4) = 2$ because $2^2 = 4$

$2^{\square}(2) = 1$ because $2^1 = 2$

$2^{\square}(\sqrt{2}) = \frac{1}{2}$ because $2^{1/2} = \sqrt{2}$

Inverse of $f(x) = 10^x$ is the function $10^{\square}(x) = \begin{pmatrix} \text{number } y \text{ for} \\ \text{which } 10^y = x \end{pmatrix}$

$10^{\square}(1000) = 3$ $\qquad$ $10^{\square}(1) = 0$

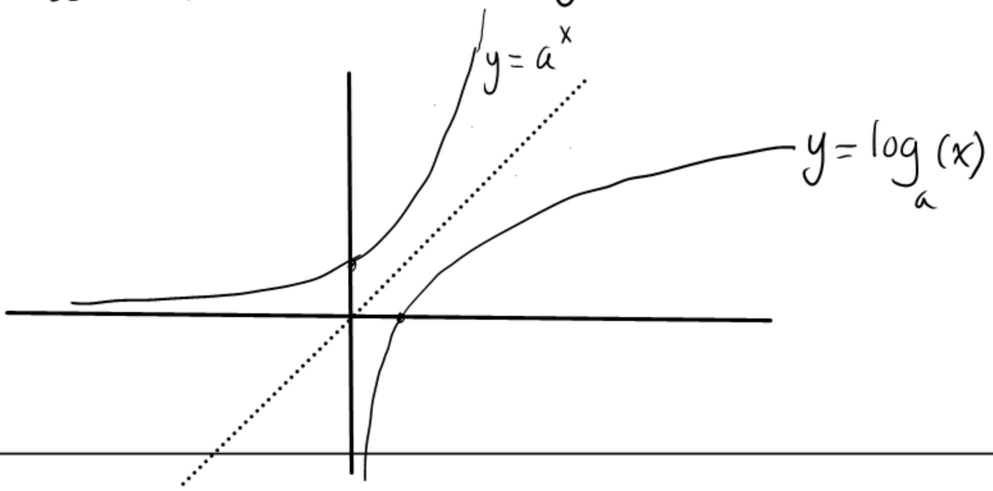
$10^{\square}(100) = 2$ $\qquad$ $10^{\square}(0.1) = -1$ (because $10^{-1} = \frac{1}{10} = 0.1$)

$10^{\square}(10) = 1$

Definition The inverse of the exponential function $f(x) = a_x$ is the function $\log_a$ (pronounced "log base a") defined as

$$\log_a(x) = a^{\square}(x) = \begin{pmatrix} \text{power } y \text{ for} \\ \text{which } a^y = x \end{pmatrix}$$

Thus $\log_a(x) = y \iff a^{\square}(x) = y \iff a^y = x$



Examples $\log_2(16) = 2^{\square}(16) = 4$ $\qquad \log_{10}\left(\frac{1}{10}\right) = 10^{\square}\left(\frac{1}{10}\right) = -1$

$\log_3(9) = 3^{\square}(9) = 2$ $\qquad \log_{10}\left(\sqrt{1000}\right) = 10^{\square}\left(1000^{\frac{1}{2}}\right) = 10^{\square}\left(10^{3/2}\right) = 3/2$.

Properties of Logarithms

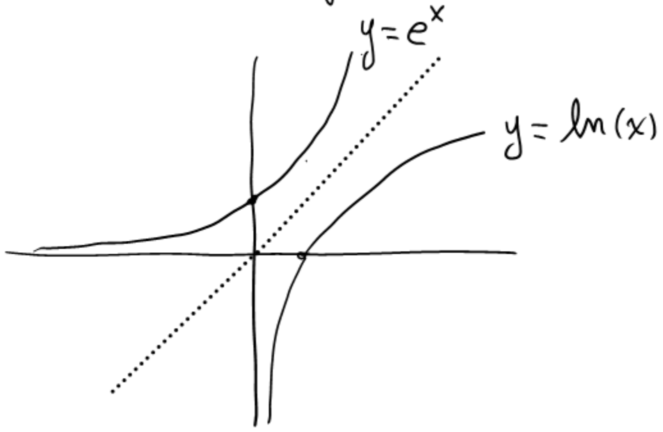
$\log_a(1) = 0$	$\log_a(xy) = \log_a(x) + \log_b(y)$
$\log_a(a) = 1$	$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$
$\log_a(a^x) = x$	$\log_a\left(\frac{1}{x}\right) = -\log_a(x)$
$a^{\log_a(x)} = x$	$\log_a(x^r) = r\log_a(x)$

In calculus, the most common base is $e = 2.718281828...$
Later we will see just why this number is so significant.

Definition

The natural exponential function is $f(x) = e^x$

The natural logarithm function is $f^{-1}(x) = \log_e(x) = \ln(x) = e^{\square}(x)$



Note:
$\ln(1) = e^{\square}(1) = 0$
$\ln(e) = e^{\square}(e) = 1$
$\ln\left(\frac{1}{e}\right) = e^{\square}\left(\frac{1}{e}\right) = -1$ etc.

Note $\ln(x)$ is a logarithm, so it obeys all log properties.

Your calculator has buttons for $\boxed{\log} = \log_{10}$

$\boxed{\ln} = \log_e$

These, combined with log laws — especially $\ln(x^r) = r\ln(x)$ — allow us to solve equations that have variable exponents:

Example Solve:

$5^{x+7} = 2^x$

$\ln(5^{x+7}) = \ln(2^x)$

$(x+7)\ln(5) = x\ln(2)$

$\to x\ln(5) + 7\ln(5) = x\ln(2)$

$x\ln(5) - x\ln(2) = -7\ln(5)$

$x(\ln(5) - \ln(2)) = -7\ln(5)$

$$\boxed{x = \frac{-7\ln(5)}{\ln(5) - \ln(2)} \approx -12.29529}$$