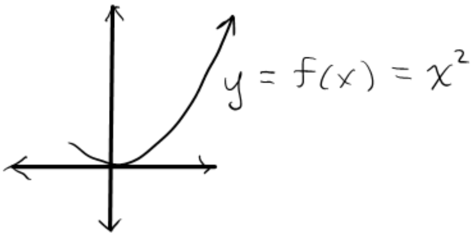
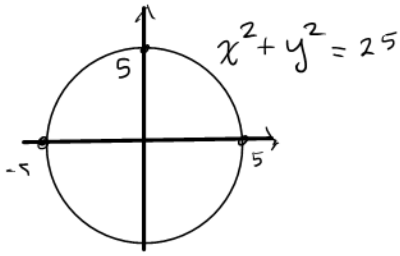
Section 3.8 Implicit Differentiation

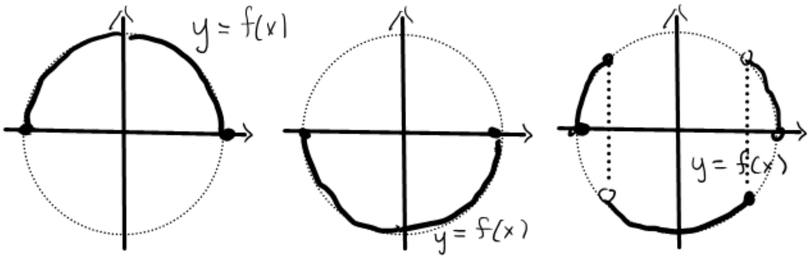
We can now differentiate just about any function $y = f(x)$



Here y is an (explicit) function of x.
We can find $f'(x)$.



But what about this situation? Here y is related to x by the equation $x^2 + y^2 = 25$, but y is not an explicit function of x.
It's not so clear what the derivative should be.



There are lots of functions $y = f(x)$ satisfying $x^2 + y^2 = 25$,

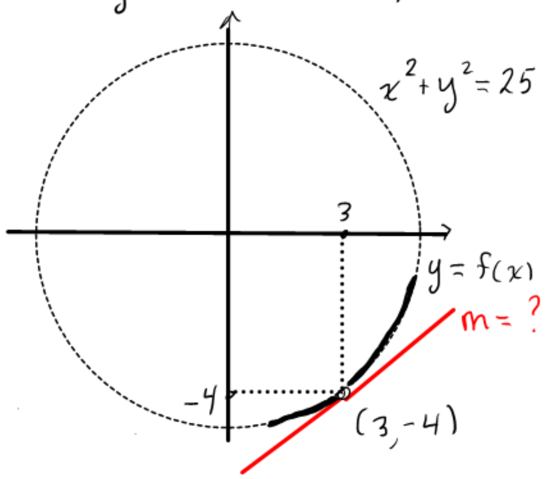
i.e. $x^2 + (f(x))^2 = 25$.

Such $f(x)$ are called implicit functions of the equation $x^2 + y^2 = 25$

Today's Goal Learn how to differentiate implicit functions.

Motivational Problem

Find the slope of the line tangent to the graph of $x^2 + y^2 = 25$ at $(3, -4)$



Let $y = f(x)$ be the above implicit function
Answer will be $m = f'(3)$

$$x^2 + y^2 = 25$$

$$x^2 + (f(x))^2 = 25$$

Now differentiate both sides

$$\frac{d}{dx}\left[x^2 + (f(x))^2\right] = \frac{d}{dx}\left[25\right]$$

Use chain rule here

$$2x + 2(f(x))^{2-1} f'(x) = 0$$

Next, solve for $f'(x)$

$$2x + 2f(x)f'(x) = 0$$

$$2f(x)f'(x) = -2x$$

$$f(x)f'(x) = -x$$

$$f'(x) = -\frac{x}{f(x)}$$

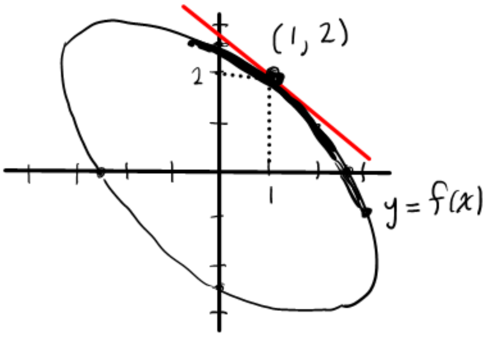
Answer: $m = f'(3) = -\frac{3}{f(3)} = -\frac{3}{-4} = \boxed{\frac{3}{4}}$

This process is called implicit differentiation.

Example

$x^2+y^2=7-xy$

Find the slope of the tangent to the graph of this equation at the point $(1,2)$.



The key to a solution is to imagine an implicit function $y=f(x)$ for the equation – one satisfying $f(1)=2$.

We seek $f'(x)$ because the slope will be $m=f'(1)$.

Think this: $y=f(x)$

$$x^2+y^2=7-xy$$

$$x^2+(f(x))^2=7-x\,f(x)$$

$$\frac{d}{dx}\left[x^2+(f(x))^2\right]=\frac{d}{dx}\left[7-x\,f(x)\right]$$

$$2x+2(f(x))^1f'(x)=0-(1)f(x)-xf'(x)$$

$$2x+2f(x)f'(x)=-f(x)-xf'(x)$$

$$2f(x)f'(x)+xf'(x)=-f(x)-2x$$

$$f'(x)(2f(x)+x)=-f(x)-2x$$

$$f'(x)=\frac{-f(x)-2x}{2f(x)+x}$$

Answer:

$$\text{Slope}=m=f'(1)=\frac{-f(1)-2\cdot 1}{2f(1)+1}$$

$$=\frac{-2-2}{2\cdot 2+1}$$

$$=\boxed{-\frac{4}{5}}$$

Write this

$$x^2+y^2=7-xy$$

$$\frac{d}{dx}\left[x^2+y^2\right]=\frac{d}{dx}\left[7-xy\right]$$

$$2x+2y\frac{dy}{dx}=0-(1)y-x\frac{dy}{dx}$$

$$2y\frac{dy}{dx}+x\frac{dy}{dx}=-y-2x$$

$$\frac{dy}{dx}(2y+x)=-y-2x$$

$$\frac{dy}{dx}=\frac{-y-2x}{2y+x}$$

Answer:

$$\text{Slope}=m=\left.\frac{dy}{dx}\right|_{(x,y)=(1,2)}=\frac{-2-2\cdot 1}{2\cdot 2+1}=\boxed{-\frac{4}{5}}$$

Both methods are essentially the same, but this one is more streamlined.

Of course it is OK to write y' instead of $\frac{dy}{dx}$

Example Find the slope of the tangent to the graph of $\sin(xy) = \cos(xy)$ at the point $(\pi, \frac{1}{4})$

Note $\sin(\pi \cdot \frac{1}{4}) = \frac{\sqrt{2}}{2} = \cos(\pi \cdot \frac{1}{4})$ so point $(\pi, \frac{1}{4})$ **is** on the graph

Solution Even though we don't know what the graph looks like, we **can** imagine an implicit function $y = f(x)$ on it with $f(\pi) = \frac{1}{4}$. So from here on out, think $y = f(x)$. That is wherever you see a y, it represents the function $y = f(x)$

$$\sin(xy) = \cos(xy)$$

$$\frac{\sin(xy)}{\cos(xy)} = \frac{\cos(xy)}{\cos(xy)}$$

$$\tan(xy) = 1$$

optional, but it can't hurt to simplify the equation first

$$\frac{d}{dx}\left[\tan(xy)\right] = \frac{d}{dx}\left[0\right]$$

$$\sec^2(xy)\left((1)y + xy'\right) = 0$$

chain rule used here. in applying it we also had to use the product rule

$$\sec^2(xy)\,y + \sec^2(xy)\,xy' = 0$$

Now solve for y' $(= \frac{dy}{dx})$ because it is what gives us the slope

$$\sec^2(xy)\,xy' = -\sec^2(xy)\,y$$

$$\frac{\sec^2(xy)\,xy'}{\sec^2(xy)\,x} = -\frac{\sec^2(xy)\,y}{\sec^2(xy)\,x}$$

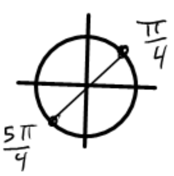
$$y' = \frac{-y}{x}$$

Answer:

$$\text{Slope} = y'\Big|_{(x,y)=(\pi,\frac{1}{4})} = \frac{-\frac{1}{4}}{\pi} = \boxed{\frac{-1}{4\pi}}$$

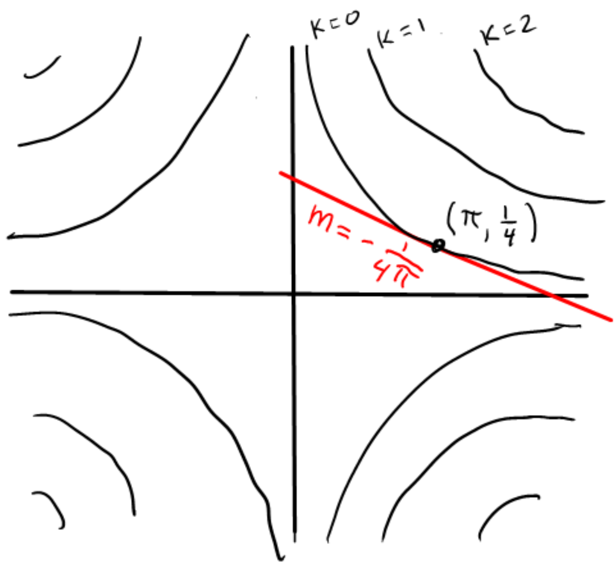
We have our answer, but let's round out our discussion by drawing the graph of $\sin(xy) = \cos(xy)$. Begin (as above) by simplifying this to

$\tan(xy) = 1$.



For this to be true we must have $xy = \frac{\pi}{4} + k\pi$ for $k = 0, \pm 1, \pm 2, \pm 3 \ldots$

Thus $y = \frac{1}{x}\left(\frac{\pi}{4} + k\pi\right)$, so the graph is a vertical scaling of graph $y = \frac{1}{x}$ by a factor of $\frac{\pi}{4} + k\pi$



Higher Order Derivatives

In the previous example we saw that if $\sin(xy) = \cos(xy)$ then $\frac{dy}{dx} = -\frac{y}{x}$

Sometimes you will be asked to find the second derivative $\frac{d^2y}{dx^2}$. To do this, just differentiate the first derivative implicitly:

$y = f(x)$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[-\frac{y}{x}\right] = -\frac{\frac{d}{dx}[y]\,x - y\frac{d}{dx}[x]}{x^2}$$

$$= -\frac{\frac{dy}{dx}x - y}{x^2}$$

Because $\frac{dy}{dx} = -\frac{y}{x}$

$$= -\frac{-\frac{y}{x}x - y}{x^2}$$

$$= -\frac{-y-y}{x^2} = \boxed{\frac{2y}{x^2}}$$

Read in text how implicit differentiation can be used to prove the power rule for rational powers:

$$\frac{d}{dx}\left[x^{\frac{p}{q}}\right] = \frac{p}{q}x^{\frac{p}{q}-1}$$

Using logarithms, the next section shows that it holds for <u>any</u> real power:

$$\frac{d}{dx}\left[x^r\right] = r\,x^{r-1}$$

<u>Example</u> $\frac{d}{dx}\left[x^{\pi}\right] = \pi x^{\pi-1}$