Section 2.6 Continuity

The notion of <u>continuity</u> is central to the theoretical development of calculus though it's hard to see why this is the case in a first course like this one. For now, let's just note that many significant theorems and results in this course will have the form

If $\boxed{f(x) \text{ is continuous,}}$ then $\boxed{\text{something significant is true}}$

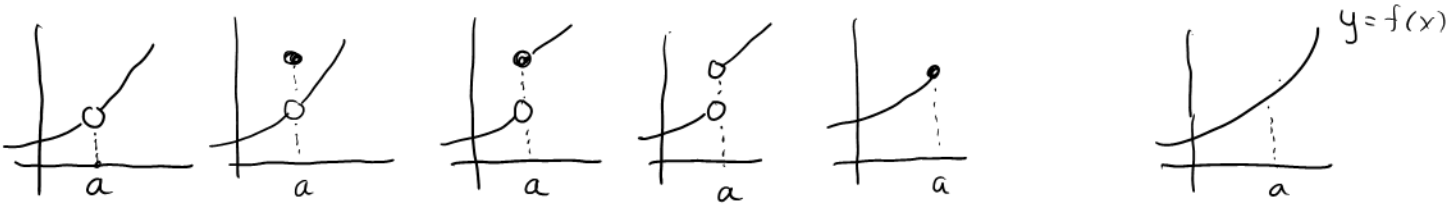
The idea of continuity is very intuitive. To motivate it, notice that the best limits of all are the simple ones that work like this:

$$\lim_{x \to a} f(x) = f(a)$$

When this happens, we say f is <u>continuous</u> at $x=a$.

> **Definition** A function f is continuous at the number $x=a$ if $\lim_{x \to a} f(x) = f(a)$. In other words, f is continuous at $x=a$ if each of the following is true.
>
> 1. $f(a)$ is defined
> 2. $\lim_{x \to a} f(x)$ exists
> 3. $\lim_{x \to a} f(x) = f(a)$

If any of these fails, $f(x)$ is <u>not</u> continuous at the point $x=a$.



①,③ fail ③ fails ②③ fail ①②③ fail ②③ fail ①②③ ALL HOLD

$f(x)$ <u>not</u> continuous at $x=a$ $f(x)$ <u>is</u> continuous at $x=a$

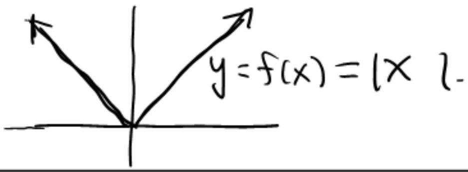
Intuitively, f being continuous at $x=a$ means that you could trace the graph with a pencil through the point $(a, f(a))$, without ever lifting your pencil.

continuous functions are "nice."

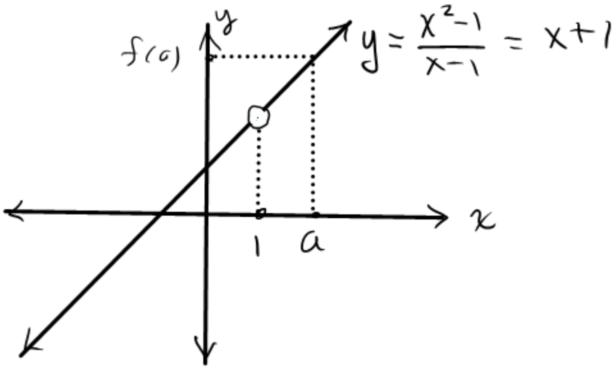
<u>Example</u> $\sin(x)$ is continuous at any a because $\lim_{x\to a} \sin(x) = \sin(a)$

<u>Example</u> A polynomial $p(x)$ is cont. at any a because $\lim_{x\to a} p(x) = p(a)$

<u>Example</u> $f(x) = |x|$ is cont. at any a because $\lim_{x\to a} |x| = |a|$.



<u>Example</u> $f(x) = \dfrac{x^2-1}{x-1}$ is <u>not</u> cont at $x=1$ because $f(1)$ undefined



But if $a \neq 1$ then:

- $\lim_{x\to a} f(x) = \lim_{x\to a} \dfrac{x^2-1}{x-1} = \lim_{x\to a} \dfrac{(x-1)(x+1)}{x-1}$
  $= \lim_{x\to a} (x+1) = \boxed{a+1}$
- Also $f(a) = \dfrac{a^2-1}{a-1} = \dfrac{(a+1)(a-1)}{a-1} = \boxed{a+1}$

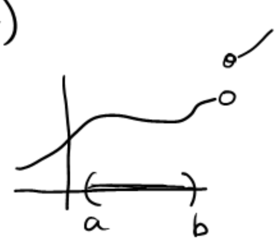
Because $\lim_{x\to a} f(x) = f(a)$, as above, f is continuous at <u>any</u> $x = a \neq 1$.

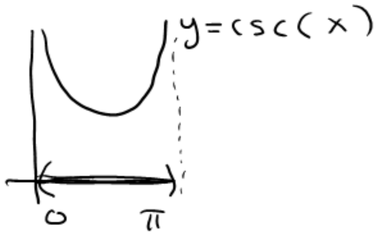
We would say $f(x)$ is continuous on the interval $(-\infty, 1)$ as well as the interval $(1, \infty)$.

We say f is continuous on an interval if it continuous at each point on the interval, i.e. it has no "jumps" on the interval

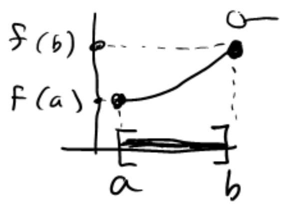
<u>Definitions</u> (Continuity on Intervals)

1. f cont. on (a,b) if its continuous at each point in (a,b)



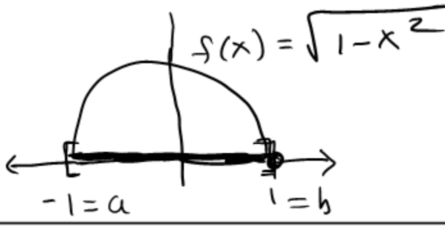


2. f cont on $[a,b]$ if it's cont. on (a,b) and $\lim_{x\to a^+} f(x) = f(a)$ and $\lim_{x\to b^-} f(x) = f(b)$.



3. f cont on $(a,b]$ if its cont. on (a,b) and $\lim_{x\to b^-} f(x) = f(b)$

4. f cont on $[a,b)$ if its cont. on (a,b) and $\lim_{x\to a^+} f(x) = f(a)$



Facts (Easy to verify with limit laws)

- Any polynomial is continuous on its domain $(-\infty, \infty)$
- Any rational function (quotient of polynomials) is cont on its domain
- $\sin(x)$ and $\cos(x)$ are cont. on their domains $(-\infty, \infty)$ because $\lim_{x \to c} \sin(x) = \sin(c)$ for any $x = c$, etc.
- $\tan(x)$, $\cot(x)$, $\sec(x)$, $\csc(x)$ are cont. on their domains
- Inverse trig functions are cont. on their domains
- Also a^x, e^x $\log_a(x)$ and $\ln(x)$.

Building up continuous functions (Theorems 2.9 and 2.11)

If $f(x)$ and $g(x)$ are continuous on their domains, then so are:

$f(x)+g(x)$ $\quad f(x)-g(x)$ $\quad f(x)g(x)$ $\quad \dfrac{f(x)}{g(x)}$ $\quad (f(x))^n$ $\quad \sqrt[n]{f(x)}$

$kf(x)$ $\quad f \circ g(x) = f(g(x))$ $\quad g \circ f(x) = g(f(x))$

k a constant

Examples The following "built up" functions are continuous on their domains because they are formed by combining continuous functions according to the above rules

$$\frac{\sin(x)}{x^2+1} \qquad \sqrt[3]{\cos(x)} \qquad \tan(x) = \frac{\sin x}{\cos x} \qquad \cos(x^2+3x+1)$$

$$\ln(x^2+1) \qquad e^{\sin(x)} \qquad \text{etc.}$$

Composition

Basic Philosophy: If $f(x)$ and $g(x)$ are continuous, then

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) \quad (\text{i.e limit goes inside } f)$$

Theorem 2.12 If $f(x)$ is cont at $b = \lim_{x \to a} g(x)$, then

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

Example

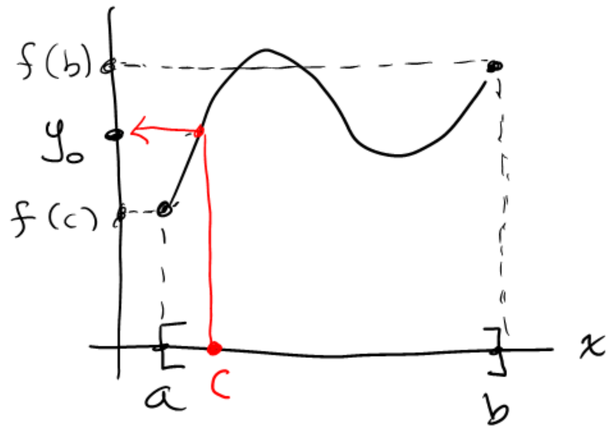
$$\lim_{x\to 2} \cos\left(\frac{\pi + x - 2}{x^2}\right) = \cos\left(\lim_{x \to 2} \frac{\pi + x - 2}{x^2}\right) = \cos\left(\frac{\pi + 2 - 2}{2^2}\right)$$

$$= \cos\left(\frac{\pi}{4}\right) = \boxed{\frac{\sqrt{2}}{2}}$$

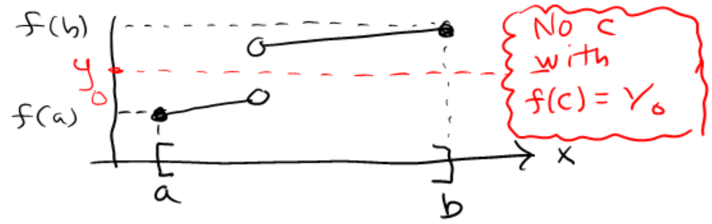
Here is a very intuitive theorem that may occasionally be useful:

Theorem 2.16 (Intermediate Value Theorem)

If $f(x)$ is continuous on a closed interval $[a,b]$ and y_0 is any number between $f(a)$ and $f(b)$, then there is a number c in $[a,b]$ for which $f(c) = y_0$.

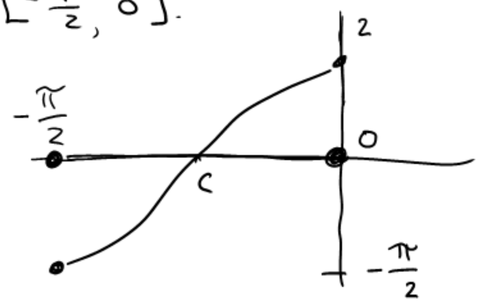


Note: continuity is essential for this to work



Example Show that the equation $x + 2\cos x = 0$ has at least one solution.

Solution Note the function $f(x) = x + 2\cos x$ is continuous on $(-\infty, \infty)$. Thus it's also cont. on $[-\frac{\pi}{2}, 0]$.



Also $f\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2} + 2\cos\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2}$

$f(0) = 0 + 2\cos(0) = 2$

Take $y_0 = 0$, so $f(-\frac{\pi}{2}) < y_0 < f(0)$.

Intermediate value theorem guarantees a c in $[-\frac{\pi}{2}, 0]$ with $f(c) = y_0 = 0$

That is, $c + 2\cos(c) = 0$, so c is a solution to $x + 2\cos(x) = 0$.