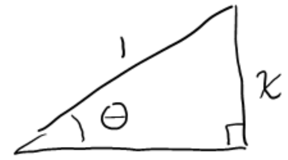
Review of Inverse Trig Functions

Trig functions and their inverses are used to relate sides of right triangles to their angles

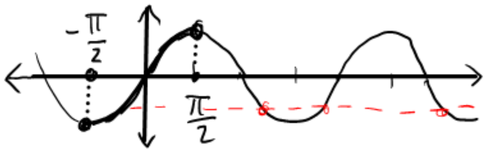


$\sin(\theta) = x$

$\sin^{-1}(x) = \theta$

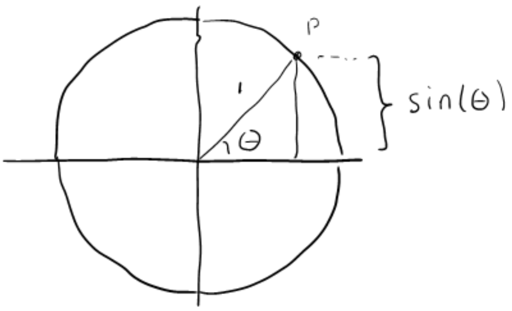
Today's Goal: Review inverse trig functions

Let's begin with $\sin(x)$. This function is not one-to-one, so it has no inverse. However, we are going to overcome this by ignoring all of its domain except $[-\frac{\pi}{2}, \frac{\pi}{2}]$.



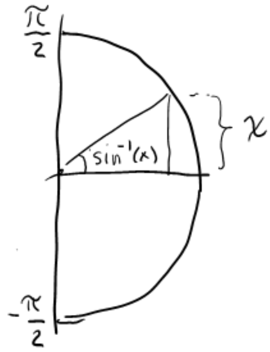
The Function $\sin^{-1}(x)$

$\sin(\theta) =$ y-coord. of P.

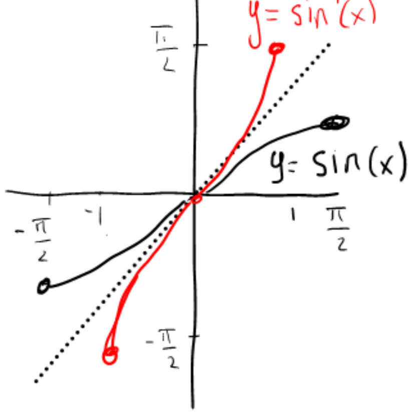


Now reverse input and output ...

$\sin^{-1}(x) = \left(\begin{array}{l}\text{angle } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ \text{for which } \sin(\theta) = x\end{array}\right)$



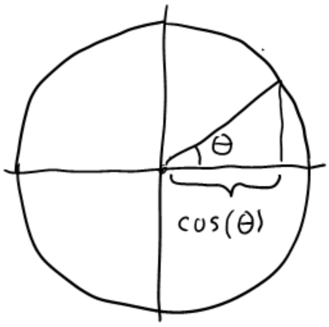
Here's its graph



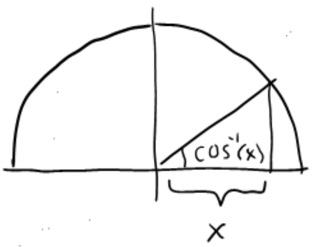
Examples $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$ $\sin^{-1}(1) = \frac{\pi}{2}$ $\sin^{-1}(-1) = -\frac{\pi}{2}$ $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

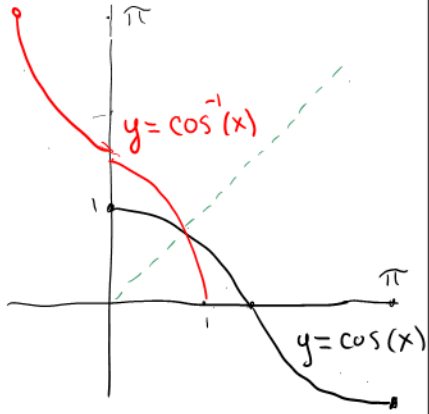
The Function $\cos^{-1}(x)$

$\cos(\theta) =$ x-coord of P



$\cos^{-1}(x) = \left(\begin{array}{l}\text{angle } \theta,\ 0 \le \theta \le \pi \\ \text{for which } \cos(\theta) = x\end{array}\right)$





Examples $\cos^{-1}(0) = \frac{\pi}{2}$ $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$ $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$ $\cos^{-1}(-1) = \pi$

Note:

$\cos(\cos^{-1}(x)) = x \leftarrow$ always true

$\cos^{-1}(\cos(x)) = x \leftarrow$ not always true!!

Example:

$\cos^{-1}(\cos(2\pi)) = \cos^{-1}(1) = 0$

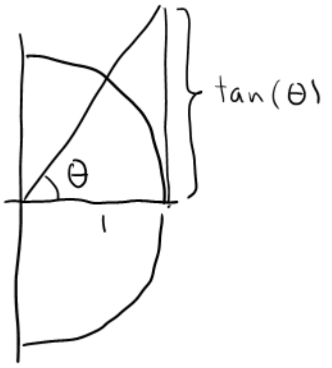
not equal

Remember:

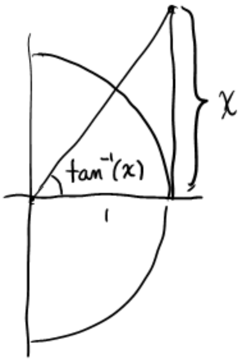
$\cos(x)$ has no inverse because it's not one-to-one. The function $\cos^{-1}(x)$ is the inverse of $\cos(x)$ with domain restricted to $[0, \pi]$

The function $\tan^{-1}$

$$\tan(\theta) = \frac{OPP}{ADJ} = OPP.$$



$$\tan^{-1}(x) = \left(\begin{array}{l}\text{angle } \theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ \text{for which } \tan(\theta) = x\end{array}\right)$$



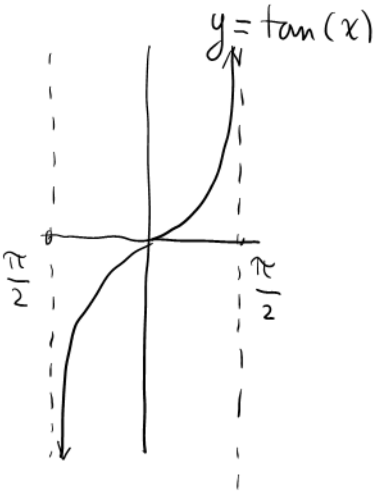
Examples

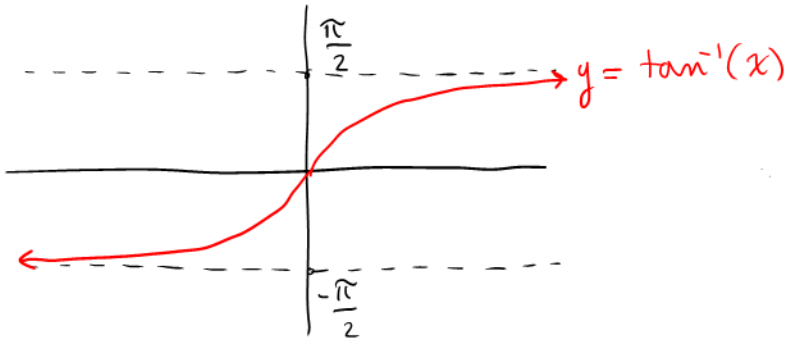
$\tan^{-1}(1) = \frac{\pi}{4}$

$\tan^{-1}(0) = 0$

$\tan^{-1}(-1) = -\frac{\pi}{4}$

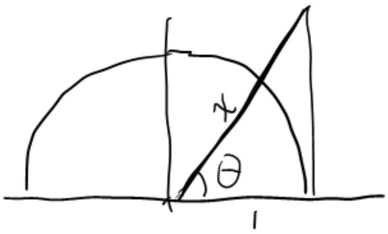
$\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$



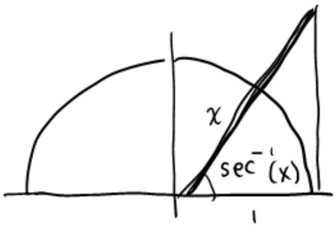


The function $\sec^{-1}(x)$

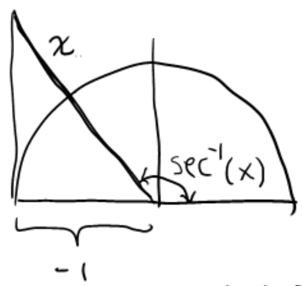
$$\sec(\theta) = \frac{HYP}{ADJ} = x$$

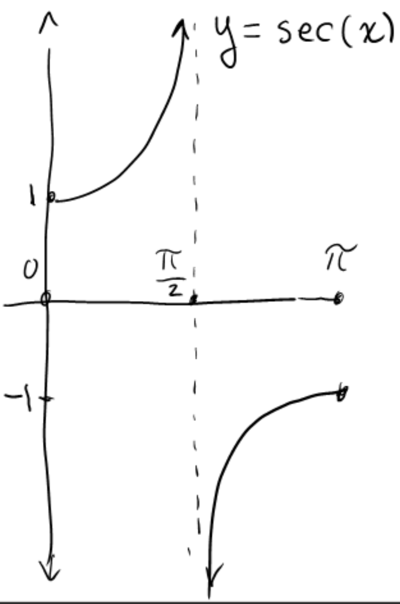


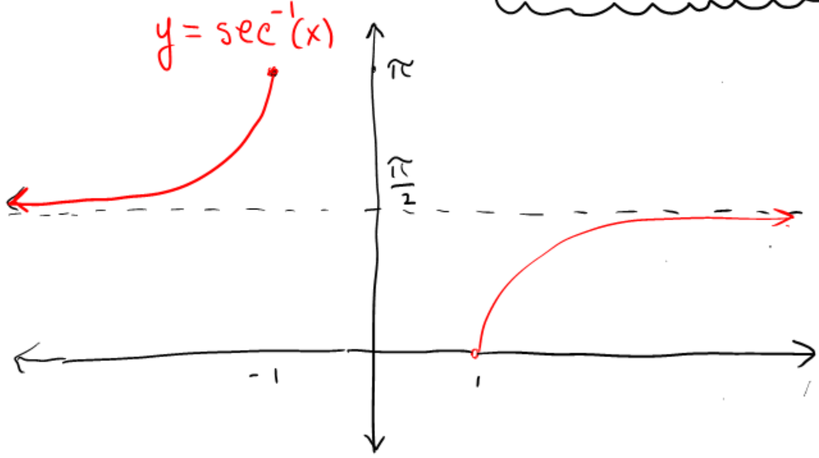
$$\sec^{-1}(x) = \left(\begin{array}{l}\text{angle } \theta,\ 0 \le \theta \le \pi \\ \text{for which } \sec(\theta) = x\end{array}\right)$$



Note when x negative picture looks this way. $\frac{HYP}{ADJ} = \frac{X}{-1} = -x.$







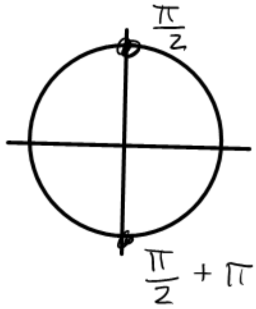
Solving Trig Equations

Example Solve $3\cos^2(x) - \cos(x) = 0$

$$\cos(x)\big(3\cos(x) - 1\big) = 0$$

Not a fimiliar angle on unit circle, so solve with $\cos^{-1}$

$\cos(x) = 0$



solutions:

$x = \frac{\pi}{2} + k\pi$

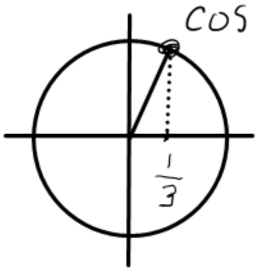
for $k = 0, \pm 1, \pm 2 \ldots$

$3\cos(x) - 1 = 0$

$3\cos(x) = 1$

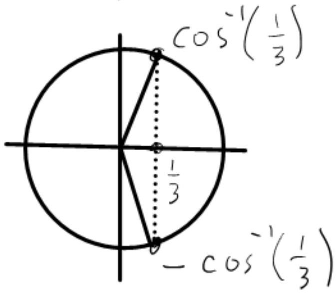
$\cos(x) = \frac{1}{3}$

$\cos^{-1}\left(\frac{1}{3}\right) = \left(\begin{array}{l}\text{angle } 0 \leq x \leq \pi \\ \text{with } \cos(x) = \frac{1}{3}\end{array}\right)$



≈ 1.23095 radians

(by calculator)



Note: $-\cos^{-1}\left(\frac{1}{3}\right)$ is also a solution to $\cos(x) = \frac{1}{3}$

Answer: solutions are

$$\begin{cases} x = \frac{\pi}{2} + k\pi \\ x = \cos^{-1}\left(\frac{1}{3}\right) + 2k\pi, \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots \\ x = -\cos^{-1}\left(\frac{1}{3}\right) + 2k\pi \end{cases}$$

Warning: Don't do it this way:

$3\cos^2(x) - \cos(x) = 0$

$3\cos^2(x) = \cos(x)$

$\frac{3\cos^2(x)}{\cos(x)} = \frac{\cos(x)}{\cos(x)}$

(dividing both sides by $\cos(x)$)

$3\cos(x) = 1$

$\cos(x) = \frac{1}{3} \Rightarrow$ solutions $\begin{cases} x = \cos^{-1}(x) + 2k\pi \\ x = \cos^{-1}(x) + 2k\pi \end{cases}$ $k = 0, \pm 1, \pm 2 \ldots$

Note In doing this we missed the solutions $x = \frac{\pi}{2} + k\pi$

Reason these are the solutions of $\cos(x) = 0$ so we accidentally divided by zero!

There are two more inverse trig functions: $\cot^{-1}(x)$ and $\csc^{-1}(x)$. Read about them in the text. It may be helpful to know that these last two inverse trig functions are rarely used.

Simplifications

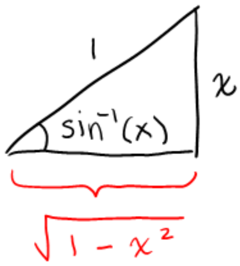
Trig functions and their inverses compose in wonderful ways. Some examples:

$$\cos(\sin^{-1}(x)) = \sqrt{1-x^2}$$

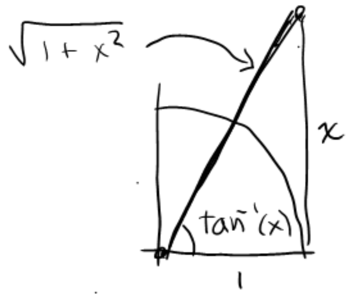
$$\tan(\sin^{-1}(x)) = \frac{OPP}{ADJ} = \frac{x}{\sqrt{1-x^2}}$$

$$\sec(\sin^{-1}(x)) = \frac{HYP}{ADJ} = \frac{1}{\sqrt{1-x^2}}$$

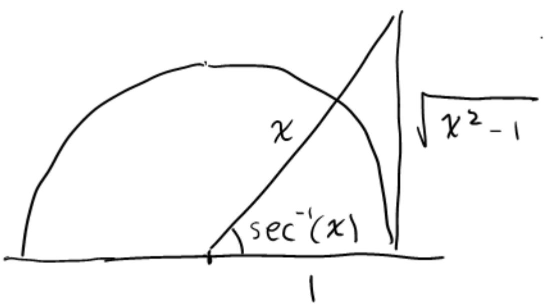
$$\sin(\sin^{-1}(x)) = x$$



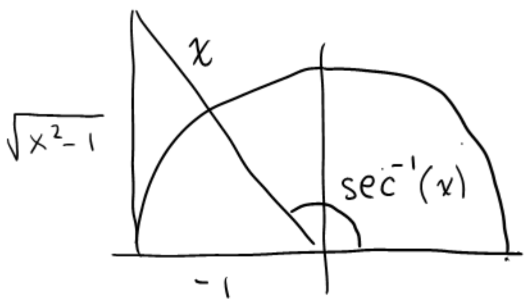
$$\sec(\tan^{-1}(x)) = \frac{HYP}{ADJ} = \sqrt{1+x^2}$$



$$\tan(\sec^{-1}(x)) = \begin{cases} \sqrt{x^2-1} & \text{if } x \geq 1 \\ -\sqrt{x^2-1} & \text{if } x \leq -1 \end{cases}$$



$x \geq 1$



$x \leq -1$

tan of this angle $\sec^{-1}(x)$ is negative