Limit Practice  
Recall the following ideas/facts:

- In evaluating \( \lim_{x \to c} \frac{f(x)}{g(x)} \), if \( g(c) = 0 \), then try to cancel the term that makes \( g(c) = 0 \).
- \( \lim_{x \to 0} \frac{\sin(x)}{x} = 1 \)
- \( \lim_{x \to 0} \frac{\cos(x) - 1}{x} = 0 \)

Examples

- \( \lim_{x \to 3} \frac{x + 3}{x^2 + 4x + 3} = \lim_{x \to -3} \frac{x + 3}{(x+3)(x+1)} = \lim_{x \to -3} \frac{1}{x+1} = \frac{1}{-3+1} = \frac{1}{2} \)

Remark:

Never do this:

Continue writing \( \lim \) until you get to a point where you can actually plug in \( x = c \) or otherwise evaluate the limit.

Points may be deducted whenever you assert two things are equal when they are not actually equal!!

Likewise, don't write your final step as, say, \( \lim_{x \to -3} = \frac{-1}{2} \)

Syntax is \( \lim_{x \to -3} f(x) = \frac{-1}{2} \)
\[
\lim_{h \to 0} \frac{\sqrt{5+h} - \sqrt{5}}{h} = \lim_{h \to 0} \frac{\sqrt{5+h} - \sqrt{5}}{h} \cdot \frac{\sqrt{5+h} + \sqrt{5}}{\sqrt{5+h} + \sqrt{5}} = \lim_{h \to 0} \frac{\sqrt{5+h}^2 - \sqrt{5}^2}{h(\sqrt{5+h} + \sqrt{5})} = \lim_{h \to 0} \frac{5+h-5}{h(\sqrt{5+h} + \sqrt{5})} = \lim_{h \to 0} \frac{1}{h(\sqrt{5+h} + \sqrt{5})} = \lim_{h \to 0} \frac{1}{\sqrt{5} + 0 + \sqrt{5}} = \frac{1}{2\sqrt{5}}
\]

\[
\lim_{x \to 0} \frac{\sin(2x)}{5x} = \frac{1}{5} \lim_{x \to 0} \frac{\sin(2x)}{x} = \frac{2}{5} \lim_{x \to 0} \frac{\sin(2x)}{2x} = \frac{2}{5} (1) = \frac{2}{5}
\]

\[
\lim_{x \to -2} \frac{\sin(x^2 + 3x + 2)}{x + 2} = \lim_{x \to -2} \frac{\sin(x^2 + 3x + 2)}{x^2 + 3x + 2} \cdot \frac{x^2 + 3x + 2}{x + 2}
\]

\[
= \lim_{x \to -2} \frac{\sin(x^2 + 3x + 2)}{x^2 + 3x + 2} \cdot \frac{(x+2)(x+1)}{x + 2}
\]

\[
= \lim_{x \to -2} \frac{\sin(x^2 + 3x + 2)}{x^2 + 3x + 2} \cdot (x+1) = (1)(-1) = -1
\]
Section 2.4  One-Sided Limits  (Continued)

Recall that to have \( \lim_{x \to c} f(x) = L \) it must be the case that \( f(x) \) approaches \( L \) no matter how \( x \) "gets to" the number \( c \). About a week ago we worked the following example.

\[ f(x) = x + \frac{|x-1|}{x-1} + 2 \]

\[ = \begin{cases} 
    x + 3 & \text{if } x < 1 \\
    x + 1 & \text{if } x > 1 
\end{cases} \]

In class we concluded \( \lim_{x \to 1} f(x) \) does not exist (DNE) because \( f(x) \) approaches different values depending on whether \( x \) approaches \( c \) from the right or the left.

Today we'll salvage some meaning from this situation, by defining right- and left-hand limits.

\[ \lim_{x \to 1^-} f(x) = 2 \quad \leftarrow \text{Left-hand limit} \]

\[ \lim_{x \to 1^+} f(x) = 4 \quad \leftarrow \text{Right-hand limit} \]

Ex  Consider \( f(x) = \sqrt{x-2} + 3 \)

\[ \lim_{x \to 2^-} f(x) \] does not exist. (Because \( f(x) \) is not even defined if \( x \) approaches 2 from left)

\[ \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \sqrt{x-2} + 3 = \sqrt{2-2} + 3 = 3 \]

\[ \lim_{x \to 2^-} f(x) \text{ DNE.} \]

Theorem \( \lim_{x \to c} f(x) = L \iff \lim_{x \to c^+} f(x) = L \quad \text{and} \quad \lim_{x \to c^-} f(x) = L \)