Section 5.4 The Fundamental Theorem of Calculus

Today we reach the apex of this course—the long promised link between antiderivatives and area.

Goal: Understand the connection between definite and indefinite integrals, i.e., the Fundamental Theorem of Calculus.

First, a tiny detail: \[ \int_a^b f(x) \, dx = \int_a^b f(t) \, dt = \int_a^b f(y) \, dy \]

The integral's value does not depend on the variable used. In this context, the variable \( x \) (or \( t \), or \( y \)) is called a dummy variable.

Also, never forget that when \( f(x) \geq 0 \) on \([a, b]\), \[ \int_a^b f(x) \, dx = \text{(area under curve)} \]

Let's start with a simple example that captures the essence of our discussion.

Motivational Example

Consider the area under the curve \( y = f(x) = x^2 + 2 \) between \( 0 \leq x \leq 2 \). Let \( A(x) = \text{(area of shaded region)} \)

\[ A(x) = \left( \text{area of } f(x) \right) + \left( \text{area of rectangle} \right) = \frac{1}{2} xx + x \cdot 2 \]

\[ \therefore A(x) = \frac{1}{2} x^2 + 2x \]

In this example (area under \( f(x) \)) = \( A(x) = \text{(antiderivative of } f(x) \))

The Fundamental Theorem of Calculus expresses exactly this link.

\( (\text{area under } f(x)) \leftrightarrow (\text{antiderivative of } f(x)) \)

We now examine this more closely.
Let's summarize our example:

\[ y = f(x) \]

\[ A(x) = \int_0^x f(t) \, dt \]

depends on \( x \)

i.e. is function of \( x \)

In Example,

\[ \frac{d}{dx} \left[ A(x) \right] = f(x) \]

i.e.

\[ \frac{d}{dx} \left[ \int_0^x f(t) \, dt \right] = f(x) \]

This illustrates

**Fundamental Theorem of Calculus Part I**

If \( f(x) \) is continuous on \([a, b]\), then the function \( A(x) = \int_a^x f(t) \, dt \)

is differentiable on \((a, b)\), and

\[ A'(x) = \frac{d}{dx} \left[ \int_a^x f(t) \, dt \right] = f(x). \]

**Proof.** Let \( A(x) = \int_a^x f(t) \, dt \)

Then \[ A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} \]

We need to show this equals \( f(x) \)

Let's analyze what's going on here with a picture.

\[ y = f(x) \]

From picture:

\[ f(x) \cdot h \leq A(x+h) - A(x) \leq f(x+h) \cdot h \]

\[ f(x) \leq \frac{A(x+h) - A(x)}{h} \leq f(x+h) \]

\[ \lim_{h \to 0} f(x) \leq \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} \leq \lim_{h \to 0} f(x+h) \]

Conclusion:

\[ A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = f(x) \]

QED
Let's follow the consequences of this. It leads to a formula for \( \int_a^x f(t) \, dt \):

\[
\frac{d}{dx} \left[ \int_a^x f(t) \, dt \right] = f(x) \quad \Rightarrow \quad \int_a^x f(t) \, dt \text{ is an antiderivative of } f(x)
\]

Let \( F(x) \) be any antiderivative of \( f(x) \).

Then \( \int_a^x f(t) \, dt = F(x) + C \)

Find \( C \) by plugging in \( x=a \): \( 0 = \int_a^a f(t) \, dt = F(a) + C \quad \Rightarrow \quad C = -F(a) \)

Thus \( \int_a^x f(t) \, dt = F(x) - F(a) \)

Letting \( x=b \): \( \int_a^b f(x) \, dx = F(b) - F(a) \)

Therefore:

\[
\int_a^b f(x) \, dx = F(b) - F(a)
\]

Conclusion

**Fundamental Theorem of Calculus Part II**

Suppose \( f(x) \) is continuous on \( [a,b] \) and \( F(x) \) is any antiderivative of \( f(x) \). Then:

\[
\int_a^b f(x) \, dx = F(b) - F(a)
\]

This is very significant because it gives a formula for \( \int_a^b f(x) \, dx \).

**Example**

Find \( \int_0^2 x^2 \, dx \)

FTC says find an antiderivative of \( f(x) = x^2 \)

\( F(x) = \frac{x^3}{3} + C \)

But FTC says any antiderivative of \( f(x) \) suffices, so let \( C=0 \)

\( F(x) = \frac{x^3}{3} \)

FTC: \( \int_0^2 x^2 \, dx = F(2) - F(0) = \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3} \)