Now that we've developed sigma notation, let's use it to find the area under a curve.

Recall \( \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \) and \( \sum_{k=1}^{n} k^2 = \frac{c}{\sum_{k=1}^{n}} f(k) \)

**Example**

Find the area under \( y = f(x) = x^2 \) between \( x = 0 \) and \( x = 2 \).

Begin by dividing \([0, 2]\) into \( n \) pieces, each of length \( \Delta x = \frac{2}{n} \).

On each interval, make a rectangle, as illustrated.

Area of rectangle \( k \) is \( f(k \frac{2}{n}) \Delta x = \left( \frac{2k^2}{n} \right) \frac{2}{n} = \frac{8k^2}{n^3} \)

\[ \left( \text{sum of areas of} \right) = \sum_{k=1}^{n} f(k \frac{2}{n}) \Delta x = \sum_{k=1}^{n} \frac{8k^2}{n^3} = \frac{8}{n^3} \sum_{k=1}^{n} k^2 = \frac{8}{n^3} \frac{n(n+1)(2n+1)}{6} = \frac{4(n+1)(2n+1)}{3} \frac{8}{n^2} = \frac{8 \cdot 2n^2 + 3n + 1}{3n^2} \]

For better approximation, make \( n \) larger.

\[ A = \lim_{n \to \infty} \left( \text{sum of areas of } n \text{ rectangles} \right) = \lim_{n \to \infty} \frac{4 \cdot 2n^2 + 3n + 1}{3n^2} = \frac{8}{3} \text{ square units} \]
Summary

How to find the area of the region under $y = f(x)$ between $a \leq b$.

1. Subdivide $[a, b]$ into $n$ equal pieces, each of length $\Delta x = \frac{b-a}{n}$.

   Get points $x_k = a + k\Delta x$ equally spaced on $[a, b]$.

2. On each subinterval $[x_{k-1}, x_k]$ establish a rectangle of height $f(x_k)$.

   Area of $k^{th}$ rectangle is $(height)(base) = f(x_k)\Delta x$

   $A \approx \sum_{k=1}^{n} f(x_k)\Delta x$

3. $A = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k)\Delta x$ ← details of working out limit depend on $f(x)$, $a$ and $b$.

A more versatile approach

Select $x_0, x_1, x_2, \ldots, x_n$ not necessarily equally spaced. This sequence is called a partition $P$ of the interval $[a, b]$.

Length of $[x_{k-1}, x_k]$ is $x_k - x_{k-1} = \Delta x_k$

Norm of $P$ is $||P|| = \text{largest } \Delta x_k$

Put a sample point $c_k$ in each $[x_{k-1}, x_k]$

For each $k$, make a rectangle with base $[x_{k-1}, x_k]$, height $f(c_k)$.

Area of rectangle $\# k$ is $f(c_k)\Delta x_k$

$A \approx \sum_{k=1}^{n} f(c_k)\Delta x_k$ ← called a Riemann Sum

$A = \lim_{||P|| \to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k$ ← area under curve
Definition: Given a function \( f(x) \) defined on \([a,b]\), the definite integral of \( f(x) \) over \([a,b]\) is the number
\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k) \Delta x \quad \text{where} \quad \Delta x = \frac{b-a}{n}, \quad x_k = a + k \Delta x
\]
(provided such limits exist).

Theorem: If \( f(x) \) is continuous on \([a,b]\), then \( \int_{a}^{b} f(x) \, dx \) exists.

*Don't worry about the proof of this. If you're really curious, take a course in Advanced Calculus (MATH 407). That's the proper place for such issues.*

Note:

**Definite Integral** \( \int_{a}^{b} f(x) \, dx \) is a number

**Indefinite Integral** \( \int f(x) \, dx \) is a function

Example:
\[
\int_{0}^{2} x^2 \, dx = \frac{8}{3} \quad \text{(By what we did earlier)}
\]
\[
\int x^2 \, dx = \frac{x^3}{3} + C
\]

The relationship between the number \( \int_{a}^{b} f(x) \, dx \) and the function \( \int f(x) \, dx \) will be spelled out by the Fundamental Theorem of Calculus, in Section 5.4.