Section 4.8 Antiderivatives

Finding the derivative of a function has been a major theme for us. Given a function $f(x)$, find its derivative $f'(x)$. That process is called differentiation.

Today we begin to look at the reverse process: Given $f'(x)$, can we find $f(x)$. This process is called antidifferentiation.

**Definition** A function $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$.

**Example** What is the antiderivative of $f(x) = 2x$?

Antiderivative: $F(x) = x^2$

Also: $F(x) = x^2 + C$

Also: $F(x) = x^2 + C$ where $C$ is any constant.

Thus $f(x) = 2x$ has infinitely many antiderivatives! Some are sketched here.
Ex. Find an antiderivative of \( f(x) = x^2 + e^{-x} - 3\sin x \).

\[
\frac{1}{3}x^3 - e^{-x} + 3\cos x \quad \text{differentiate} \quad x^2 + e^{-x} - 3\sin x
\]

Antiderivative is \( F(x) = \frac{1}{3}x^3 - e^{-x} + 3\cos x \).

In general, \( F(x) = \frac{1}{3}x^3 - e^{-x} + 3\cos x + C \).

**Definition** If \( f(x) \) has an antiderivative \( F(x) \) (i.e., if \( F'(x) = f(x) \)), then \( F(x) + C \) is called the **most general antiderivative** of \( f(x) \) or the **indefinite integral** of \( f(x) \).

Note: \( F(x) + C \) stands for infinitely many functions, one for each possible value of \( C \). Thus \( F(x) + C \) is the set of all antiderivatives of \( f(x) \).

**Notation** The indefinite integral of \( f(x) \) is denoted as

\[
\int f(x) \, dx = F(x) + C
\]

Thus \( \int f(x) \, dx \) stands for all antiderivatives of \( f(x) \).

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**Ex.** \[
\int x^3 \, dx = \frac{1}{4}x^4 + C
\]

**Ex.** \[
\int (x^2 + e^{-x} - 3\sin x) \, dx = \frac{1}{3}x^3 - e^{-x} + 3\cos x + C
\]

**Ex.** \[
\int \frac{2x}{1 + x^2} \, dx = \ln(1 + x^2) + C
\]

**Ex.** \[
\int \frac{1}{1 + x^2} \, dx = \tan^{-1} x + C
\]

**Ex.** \[
\int \frac{1}{x} \, dx = \ln |x| + C
\]

On page 179, text shows

\[
\frac{d}{dx} [\ln |x|] = \frac{1}{x}
\]

You want to say this is \( \ln x \), but that's not quite right, because \( \ln x \) is defined only for positive \( x \), so its derivative \( \frac{1}{x} \) is interpreted to have domain \((0, \infty)\).

However, here we might think of \( \frac{1}{x} \) with domain \((-\infty, 0) \cup (0, \infty)\).

Thus \( \int \frac{1}{x} \, dx = \ln |x| + C \).
Never Forget:
\[ \int f(x) \, dx = F(x) + C \iff \frac{d}{dx} [F(x) + C] = f(x) \]

Any derivative formula, run in reverse, becomes an antiderivative formula. Here are the main ones:

\[ \int k \, dx = kx + C \]
\[ \int x^n \, dx = \frac{1}{n+1} x^{n+1} + C \quad \text{Provided } n \neq -1 \]
\[ \int x^{-1} \, dx = \ln|x| + C \quad \text{Here's what you do when } n = 1. \]
\[ \int \sin x \, dx = -\cos x + C \]
\[ \int \cos x \, dx = \sin x + C \]
\[ \int \sec^2 x \, dx = \tan x + C \]
\[ \int \csc^2 x \, dx = -\cot x + C \]
\[ \int \sec x \tan x \, dx = \sec x + C \]
\[ \int \csc x \cot x \, dx = -\csc x + C \]
\[ \int e^x \, dx = e^x + C \]
\[ \int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C \]
\[ \int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C \]
\[ \int \frac{1}{x\sqrt{x^2-1}} \, dx = \sec^{-1} x + C \]
\[ \int a^x \, dx = \frac{1}{\ln(a)} \cdot a^x + C \]
\[ \int k \, f(x) \, dx = k \int f(x) \, dx \]
\[ \int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx. \]

Summary:
We now have two opposite operations on functions:
\[
\begin{align*}
\frac{d}{dx} [F(x) + C] & \rightarrow F(x) + C \\
\int f(x) \, dx & \rightarrow \frac{d}{dx} [F(x) + C]
\end{align*}
\]

More about all this next time!