Section 4.2 The Mean Value Theorem

The Mean Value theorem is a significant theoretical result. Although you will not use it often, it lays the foundation for Chapter 5, and is used in proofs of many fundamental results. We first address a preliminary result called Rolle’s Theorem.

**Rolle’s Theorem:** Suppose a function \( f(x) \) is continuous on an interval \([a, b]\) and differentiable on the interval \((a, b)\). (That is, \( f(x) \) exists and is defined at every number \( x \in (a, b) \).) If \( f(a) = f(b) \) then there exists at least one number \( c \in (a, b) \) for which \( f'(c) = 0 \).

\[
\begin{align*}
f(a) &= f(b) \\
\text{ Rolle's Theorem guarantees } & \text{ there is a number with } \\
& \text{ } f'(c) = 0, \text{ as illustrated.} \\
& \text{ Note that there may be several such } c, \text{ as is the case here.}
\end{align*}
\]

The text gives a careful proof of Rolle’s theorem, which you should read. But notice the Theorem is very intuitive.

Move a horizontal line down (or up) until it hits a point on the graph of \( y = f(x) \). The point of first contact has an \( x \)-coordinate \( c \) for which \( f'(c) = 0 \), as illustrated here.

From Rolle’s Theorem we get:

**The Mean Value Theorem:** Suppose \( f(x) \) is continuous on \([a, b]\) and differentiable on \((a, b)\) then there is a \( c \in (a, b) \) for which

\[
f'(c) = \frac{f(b) - f(a)}{b - a}.
\]

\[
\begin{align*}
slope &= f'(c) \\
\text{ and } \quad y &= f(x) \\
\text{ slope } &= \frac{f(b) - f(a)}{b - a} \\
\text{ MVT says there is at least one } c \text{ in } (a, b) \\
&\text{ with } f'(c) = \frac{f(b) - f(a)}{b - a} \text{ i.e. the two slopes are equal.}
\end{align*}
\]

Read the proof of the MVT in the text, but also note and appreciate that the theorem is very intuitive and common-senseical.
Note that there can be several \( c \) in \((a, b)\) with
\[
\xi'(c) = \frac{f(b) - f(a)}{b - a}.
\]
MVT says there is at least one such \( c \).

**Example** (A way of thinking about the MVT.)

Suppose you drive 30 miles in 20 minutes. (\(\frac{1}{3}\) hour). Did you break the speed limit? Your intuition says **YES** because your average velocity is \(\frac{30 \text{ mi}}{\frac{1}{3} \text{ hour}} = 90 \text{ mph}\). The mean value confirms this.

Say your position at time \( t \) is \( s(t) \).

\[
\begin{array}{c}
\xrightarrow{t = 0} s(x) \xrightarrow{t = \frac{1}{3}} \text{30 mi} \\
\end{array}
\]

MVT says at some time \( t = c \),
\[
s'(c) = \frac{s\left(\frac{1}{3}\right) - s(0)}{\frac{1}{3} - 0} = \frac{30}{\frac{1}{3}} = 90 \text{ mph}
\]

MVT simply says that at some instant \( t = c \), your instantaneous velocity equals your average velocity.

**Mathematical Consequences of MVT.**

**Corollary 1** Suppose \( f'(x) = 0 \) on an interval \((a, b)\).
Then \( f(x) = C \) on \((a, b)\), where \( C \) is a constant.

**Proof** Take \( x \in (a, b) \). By MVT, there exists a \( c \in [a, x] \) with
\[
0 = f'(c) = \frac{f(x) - f(a)}{x-a}
\]
\[
\Rightarrow 0(a-x) = f(x) - f(a), \text{ i.e., } 0 = f(x) - f(a). \text{ Then } f(x) = -f(a) = C.
\]

**Corollary 2** Suppose \( f'(x) = g'(x) \) on \((a, b)\).
Then \( f(x) = g(x) + C \) for some constant \( C \).

**Proof** If \( f'(x) = g'(x) \) on \((a, b)\), then \( f(x) - g(x) = 0 \) on \((a, b)\). That is, \( (f+g)'(x) = 0 \) on \((a, b)\). By corollary 1,
\[
(f+g)x = C, \text{ i.e., } f(x) + g(x) = C
\]
Thus \( f(x) = g(x) + C \).

**Corollary** says that if \( f'(x) = g'(x) \), i.e., slopes of \( f(x) \) and \( g(x) \) agree then \( f(x) = g(x) + C \).