**Section 4.4  Concavity and Curve Sketching**

**Goal:** Use derivatives to help us understand graphs of functions, and to better understand local/absolute max/min.

**Basic Theme:** What do $f'(x)$ and $f''(x)$ tell us about $f(x)$?

*Recall* $f'(x)$ tells increase/decrease of $f(x)$:
- If $f'(x) > 0$ on $(a, b)$ then $f(x)$ increases on $(a, b)$
- If $f'(x) < 0$ on $(a, b)$ then $f(x)$ decreases on $(a, b)$

**Concavity**

In addition to increasing/decreasing $f(x)$ can also exhibit certain kinds of concavity

The concavity of $f(x)$ can be up or down at different parts of the graph:
- $f(x)$ concave down on $(-\infty, 1)$
- $f(x)$ concave up on $(1, 2)$
- $f(x)$ concave down on $(2, \infty)$

**Definition:** A point $(a, f(a))$ on the graph of $y = f(x)$ where the concavity changes is called a point of inflection or an inflection point.

[Text requires also that $f'(a)$ exists, but that's splitting hairs]

There are 4 possible combinations of increase/decrease and concave up/down:
How can you tell where \( f(x) \) is concave up or down?

\[
f(x) \text{ concave up} \quad y = f'(x)
\]
\[
f(x) \text{ concave down} \quad y = f'(x)
\]

\( f(x) \) increasing

Thus \( f''(x) > 0 \)

\( f'(x) \) decreasing

Thus \( f''(x) < 0 \)

**Conclusion**
- If \( f''(x) > 0 \) on \((a, b)\), then \( f(x) \) is **concave up** on \((a, b)\).
- If \( f''(x) < 0 \) on \((a, b)\), then \( f(x) \) is **concave down** on \((a, b)\).

**Example** Find where \( f(x) = \ln(x^2 + 1) \) is increasing / decreasing and concave up / down.

**Increasing / decreasing**

\[
f'(x) = \frac{2x}{x^2 + 1}
\]

**Concavity**

\[
f''(x) = \frac{2(x^2 + 1) - 2x \cdot 2x}{(x^2 + 1)^2} = \frac{2x^2 + 2 - 4x^2}{(x^2 + 1)^2} = \frac{2(1 - x^2)}{(x^2 + 1)^2} = \frac{2(1-x)(1+x)}{(x^2 + 1)^2}
\]

**Note:**

Abs min at \( x = 0 \)

**Graphical Representation:**

- **I.P.,** \((-1, \ln(2))\)
- **I.P.,** \((1, \ln(2))\)

**Interval Analysis:**

- **Concave down**
  - Decreasing
  - \(- \infty < x < -1\)

- **Concave up**
  - Increasing
  - \(-1 < x < 1\)

- **Concave down**
  - \(1 < x < \infty\)
Notice that these methods allow us to quickly sketch a good graph of $y = f(x)$. You should get some practice sketching graphs this way. Be sure to work some exercises.

For today, we will do one more example.

**Example**  Graph of derivative $f'(x)$ of a function $f(x)$ is shown.
Find where $f(x)$ increases / decreases
Find where $f(x)$ is concave up / down.

Which of the following is true?

- $f(0) > f(1)$
- $f(0) < f(1)$
- $f(0) = f(1)$

\[ y = f(x) \]
\[ y = f'(x) \]

\[
\begin{array}{ccccccc}
- & - & + & + & - & - & +
\end{array}
\quad
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+ & + & + & + & + & + & +
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\quad
\begin{array}{ccccccc}
f'(x)
\end{array}
\quad
\begin{array}{ccccccc}
f''(x)
\end{array}
\]

**Answers:**
- $f(x)$ increases on $(1, 2)$ and $(4, \infty)$
- $f(x)$ decreases on $(-\infty, 1)$ and $(2, 4)$
- $f(x)$ concave up on $(-\infty, 1.5)$ and $(3, \infty)$
- $f(x)$ concave down on $(1.5, 3)$

Since $f(x)$ decreases between $0$ and $1$, we must have $f(0) > f(1)$. 