Section 4.3 The First Derivative Test

Goals
1. Find where \( f(x) \) increases/decreases
2. Locate local extrema of \( f(x) \)

1. Facts let \( I = (a, b) \) be an interval.
   - If \( f'(x) > 0 \) on \( I \) then \( f(x) \) increases on \( I \)
   - If \( f'(x) < 0 \) on \( I \) then \( f(x) \) decreases on \( I \)

Example
Find intervals on which \( f(x) = 2x^3 - 3x^2 + 5 \) increases/decreases

\[
f'(x) = 6x^2 - 6x = 6x(x - 1)
\]

After factoring \( f'(x) \), we can analyze its signs on various intervals:

\[
\begin{array}{c|cc}
& 0 & 1 \\
\hline
f'(x) & - & + \\
\end{array}
\]

Conclusions
- \( f(x) \) increases on \((- \infty, 0) \) and \((1, \infty) \)
- \( f(x) \) decreases on \((0, 1) \)

2. Now let's turn to our second goal: Finding local extrema.

The above example serves as a guide. Notice that:

- Local extrema happen at critical points. \( \{0 \) and \( 1 \) above\}
- Local max happens when \( f(x) \) stops increasing, starts decreasing
- Local min happens when \( f(x) \) stops decreasing, start increasing

\[
y = f(x)
\]

---

+++ | -- | +++ | +++ | -- | +++ | -- | f(x)
Conclusion

First Derivative Test (for finding local extrema)

Suppose \( c \) is a critical point of \( f(x) \)
(i.e., \( c \) is in the domain of \( f(x) \) and \( f'(c) = 0 \) or \( f'(c) \) is undefined)
1. If \( f'(x) \) changes from + to - at \( c \), then \( f(x) \) has a local max at \( c \).
2. If \( f'(x) \) changes from - to + at \( c \), then \( f(x) \) has a local min at \( c \).
3. If \( f'(x) \) does not change sign at \( c \), the no local extremum at \( c \).

How to find the local extrema of \( f(x) \)

A. Find the critical points.
B. Apply 1st derivative test.

Example Find local extrema of \( f(x) = x e^x \) on \( (-\infty, \infty) \)

A. \( f'(x) = (1)e^x + xe^x = e^x(1+x) = 0 \)

Only one critical point: \( c = -1 \)

Local minimum at \( x = -1 \)
No local max

Example Find local extrema of \( f(x) = \frac{3}{2}x^2 - \frac{2}{3}x \)

\( f'(x) = \frac{2}{3}x^3 - \frac{2}{3} = \frac{2}{3} \left( \frac{3}{x} - \frac{1}{3} \right) \)

f'(0) undefined \( \rightarrow \) critical points: 0 and 1.

Local max at \( x = 1 \)
Local min at \( x = 0 \)
Example. Find local extrema of \( f(x) = \frac{1}{x^2} = x^{-2} \)

\[ f'(x) = -2x^{-3} = -\frac{2}{x^3} \]

\( y = f(x) \)

++ + + + - - - - - \( f'(x) \)

Can't have a local extrema at 0 if \( f(0) \) is undefined.

No local extrema.

There are no critical points.

\( x = 0 \) is not in the domain of \( f(x) \), so it is not a critical point.

Example. Find local extrema of \( f(x) = x + \sin x \)

\[ f'(x) = 1 + \cos x \]

Critical points are those \( x = c \) for which \( f'(x) = 1 + \cos x = 0 \)

[Diagram showing critical points]

Critical pts. \( c = \pi + k 2\pi \)

\[ = \pi (1 + 2k) \]

\[ = \pi \text{m for odd m} \]

Happens when \( \cos x = -1 \), \( \text{i.e. } x = \pi + k 2\pi \) for \( k = 0, \pm 1, \pm 2, ... \)

No relative extrema.