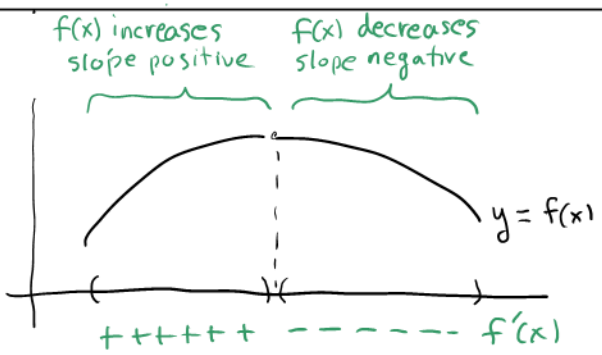
Section 4.3 The First Derivative Test

Goals ① Find where $f(x)$ increases / decreases
② Locate local extrema of $f(x)$

①

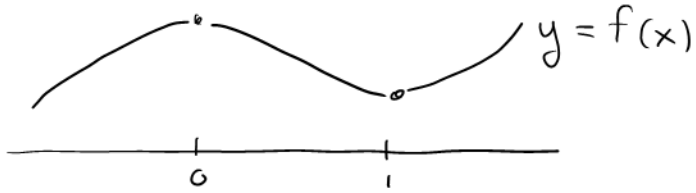
Facts Let $I = (a, b)$ be an interval.

- If $f'(x) > 0$ on I then $f(x)$ increases on I
- If $f'(x) < 0$ on I then $f(x)$ decreases on I



Example Find intervals on which $f(x) = 2x^3 - 3x^2 + 5$ increases/decreases

$$f'(x) = 6x^2 - 6x$$
$$= 6x(x-1)$$



After factoring $f'(x)$ we can analyze its signs on various interval:

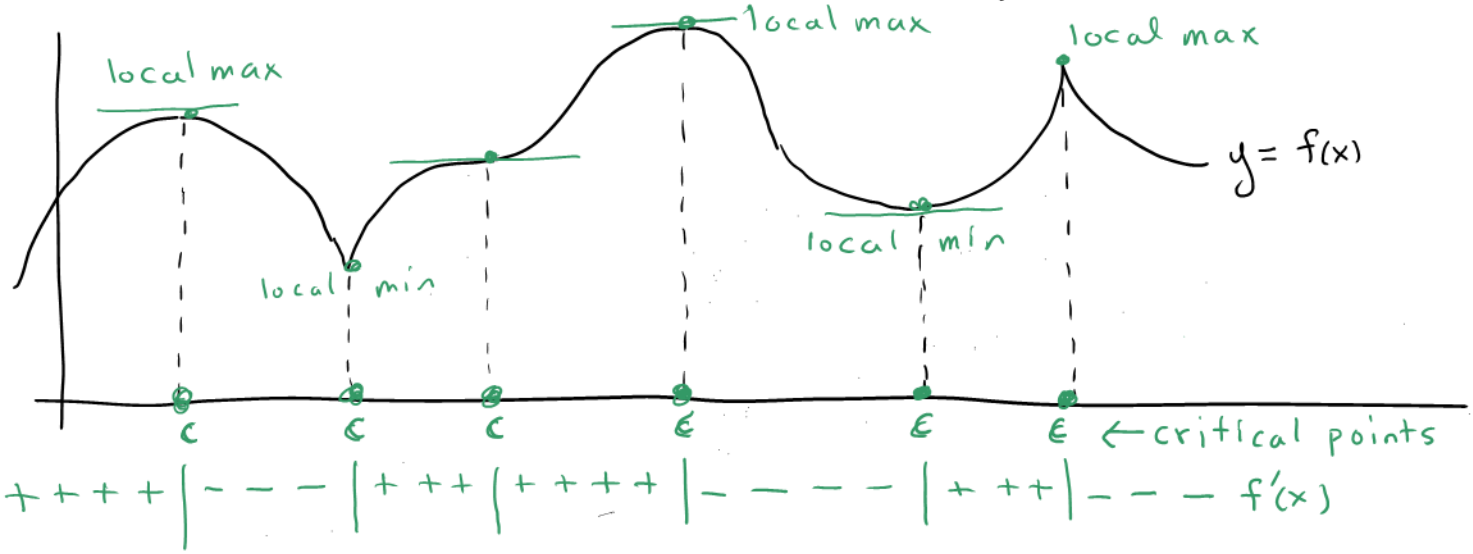
− − − −	+ + + +	+ + + +	$6x$
− − − −	− − − −	+ + + +	$(x-1)$
+ + + +	− − − −	+ + +	$f'(x) = 6x(x-1)$

Conclusion $f(x)$ increases on $(-\infty, 0)$ and $(1, \infty)$
$f(x)$ decreases on $(0, 1)$

② Now let's turn to our second goal: Finding local extrema.

The above example serves as a guide. Notice that:

- Local extrema happen at critical points. {0 and 1 above}
- Local max happens where $f(x)$ stops increasing, starts decreasing
- Local min happens where $f(x)$ stops decreasing, starts increasing



Conclusion

First Derivative Test (for finding local extrema)

Suppose c is a critical point of $f(x)$
(i.e. c is in the domain of $f(x)$ and $f'(c)=0$ or $f'(c)$ is undefined)

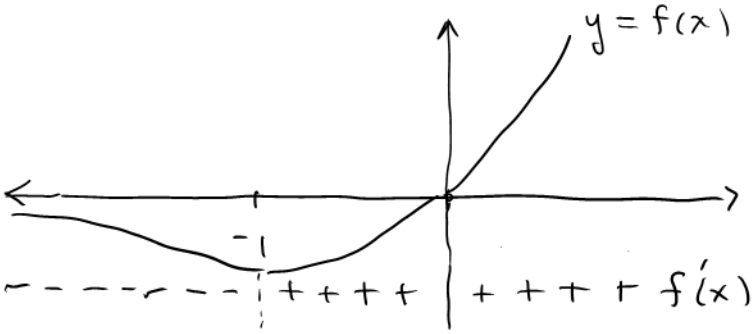
1. If $f'(x)$ changes from $+$ to $-$ at c, then $f(x)$ has a local max at c.
2. If $f'(x)$ changes from $-$ to $+$ at c, then $f(x)$ has a local min at c.
3. If $f'(x)$ does not change sign at c, the no local extremum at c.

How to find the local extrema of $f(x)$
- (A) Find the critical points.
- (B) Apply 1st derivative test

Example Find local extrema of $f(x) = xe^x$ on $(-\infty, \infty)$

(A) $f'(x) = (1)e^x + xe^x = e^x(1+x) = 0$

Only one critical point: $c = -1$

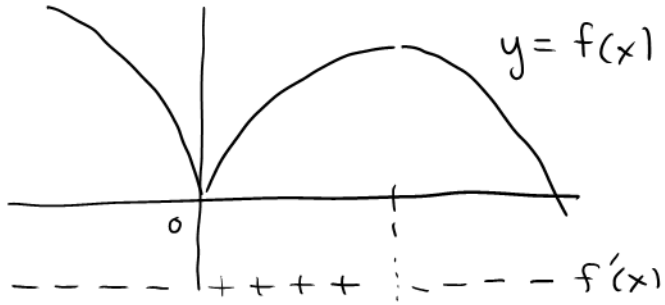


Local minimum at $x = -1$
No local max

Example Find local extrema of $f(x) = \sqrt[3]{x}^{\,2} - \frac{2}{3}x = x^{\frac{2}{3}} - \frac{2}{3}x$

$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} - \frac{2}{3} = \frac{2}{3\sqrt[3]{x}} - \frac{2}{3} = \frac{2}{3}\left(\frac{1}{\sqrt[3]{x}} - 1\right)$

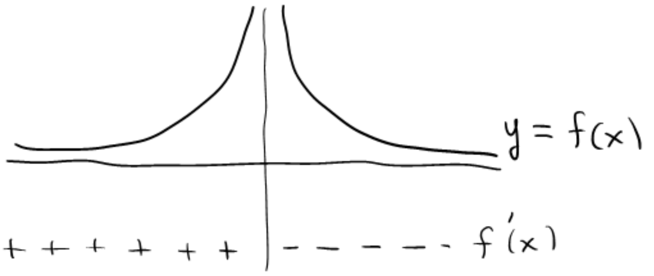
$\left.\begin{array}{l} f'(0) \text{ undefined} \\ f'(1) = 0 \end{array}\right\}$ critical points: 0 and 1.



Local max at $x = 1$
Local min at $x = 0$

Example Find local extrema of $f(x) = \frac{1}{x^2} = x^{-2}$

$f'(x) = -2x^{-3} = \frac{-1}{x^3}$



No local extrema!

There are no critical points.

x = 0 is not in the domain of of f(x) so it is not a critical point

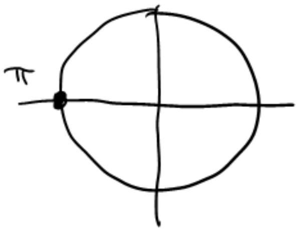
Can't have a local extrema at 0 if f(0) is undefined.

Example Find local extrema of $f(x) = x + \sin x$

$f'(x) = 1 + \cos x$

Critical points are those $x = c$ for which $f'(x) = \underbrace{1 + \cos x = 0}$

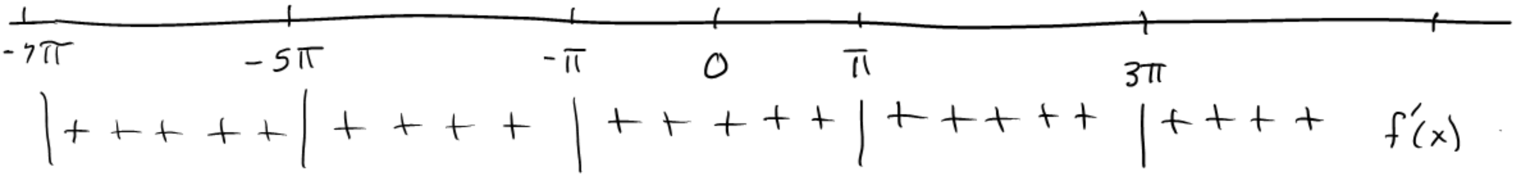
Happens when $\cos x = -1$, i.e $x = \pi + k2\pi$ for $k = 0, \pm 1, \pm 2 \ldots$



Critical pts. $c = \pi + k2\pi$

$= \pi(1 + 2k)$

$= \pi m$ for odd m



No relative extrema