Chapter 4 Applications of Derivatives

Now that we know how to find derivatives and what they mean, we turn to some applications. The main topic of Chapter 4 involves finding maximum and minimum values of functions.

Examples

Profit = \( P(x) \) \(<\) Goal find \( x \) that maximizes this.

Cost = \( C(x) \) \(<\) Goal find \( x \) that minimizes this.

In general, if \( f(x) \) is a good or desirable quantity, we want to find an \( x \) that maximizes \( f(x) \) (i.e., makes it as large as possible). If \( f(x) \) is a bad or undesirable quantity, we seek an \( x \) that minimizes it (i.e., makes it as small as possible). Thus our goal is to find an optimal outcome. Derivatives (as we will see) provide a means of doing this. Section 4.1 sets up some main ideas and definitions.

Section 4.1 Extreme Values of Functions

Definitions

Suppose \( f(x) \) is defined on a domain \( D \) (usually an interval).

- \( f(x) \) has an absolute maximum at \( c \) in \( D \) if \( f(x) \leq f(c) \) for all \( x \) in \( D \).
- \( f(x) \) has an absolute minimum at \( c \) in \( D \) if \( f(x) \geq f(c) \) for all \( x \) in \( D \).

Examples

\[
\begin{align*}
\text{f(x) has absolute maximum at } &c \text{ because } f(x) \leq f(c) \text{ for all } x. \\
\text{f(x) has absolute minimum at } &d \text{ because } f(x) \geq f(d) \text{ for all } x.
\end{align*}
\]

This function has an absolute maximum of \( f(c) \), which is attained at two separate values of \( c \).

The absolute minimum occurs at \( d \), an endpoint of \( D \).

Terminology:

If \( f(x) \) has an abs. max at \( c \), \( f(c) \) is an absolute maximum of \( f(x) \) at \( c \).

If \( f(x) \) has an abs. min. at \( d \), \( f(d) \) is an absolute minimum of \( f(x) \) at \( d \).

\( f(c) \) and \( f(d) \) are absolute extrema.
Examples

\[ y = f(x) \]

- \( D = (-\infty, \infty) = \mathbb{R} \)
  - Absolute minimum at \( c \)
  - No absolute maximum

- \( D = [-2, 3] \)
  - Absolute maximum at -2
  - Absolute minimum at 1

- \( D = (-\infty, \infty) = \mathbb{R} \)
  - No absolute max.
  - No absolute min.

So in some circumstances absolute extrema don't exist. But there is one situation in which it is guaranteed:

**Theorem (Extreme Value Theorem)**

If \( f(x) \) is continuous on a closed interval \([a, b]\), then it has both an absolute max. and an absolute min. on \([a, b]\).

![Graph of a function](image)

Intuitively this seems plausible, almost obvious, but the proof is subtle and is left to more advanced courses.

Note that the condition of continuity is essential, as this picture suggests:

![Graph with no extrema](image)

Our goal is to locate extreme values. Where do they occur? Our pictures suggest that extreme values can happen at endpoints and values of \( c \) where \( f'(c) = 0 \).

Where else? An example is instructive:

Consider \( f(x) = \sqrt{x^2} = x^{\frac{2}{3}} \)

\[ f'(x) = \frac{2}{3} x^{\frac{2}{3} - 1} = \frac{2}{3} x^{-\frac{1}{3}} = \frac{2}{3} \frac{1}{\sqrt[3]{x}} = \frac{2}{3\sqrt[3]{x}} \]

Note that \( f(x) = \sqrt{x^2} \) has an absolute minimum at \( c = 0 \) and \( f'(0) \) is not defined.

Conclusion: Extreme values can occur at endpoints or values \( c \) for which \( f(c) = 0 \) or \( f'(c) \) is undefined.
This leads to a definition

**Definition** A number \( c \) in the domain of \( f(x) \) is called a critical point if \( f'(c) = 0 \) or \( f'(c) \) is undefined.

Note critical points are candidates for extreme values.

This leads to our main result of the day:

**How to find the extreme values of \( f(x) \) on a closed interval \([a, b]\):**

1. Find all critical points \( c \) in \([a, b]\).
2. Evaluate each \( f(c) \) and \( f(a) \) and \( f(b) \).
3. Select the largest value from Step(2). It’s the abs. max.
   Select the smallest value from Step(2). It’s the abs. min.

**Example** Find absolute extrema of \( f(x) = \sqrt[3]{x^2 - \frac{4}{3}x} \) on \([0, 27]\)

(1) Find critical pts. of \( f(x) \)
   To do this, examine \( f'(x) = \frac{2}{3\sqrt[3]{x}} - \frac{4}{3} \)
   Note \( f'(0) \) is undefined, so \( 0 \) is a critical point.
   The other critical points will be those \( x = c \) for which \( f'(c) = 0 \).
   To find them, set \( f'(x) = 0 \) and solve for \( x \).

   \[
   \frac{2}{3\sqrt[3]{x}} - \frac{4}{3} = 0 \\
   \frac{2}{3\sqrt[3]{x}} = \frac{4}{3} \\
   \frac{2}{3} \left( \frac{1}{\sqrt[3]{x}} - 2 \right) = 0 \\
   \frac{1}{\sqrt[3]{x}} - 2 = 0 \\
   \frac{1}{\sqrt[3]{x}} = 2 \\
   \sqrt[3]{x} = \frac{1}{2} \\
   x = \left( \frac{1}{2} \right)^3 = \frac{1}{8}
   \]

   Thus the critical points are \( c = 0 \) and \( c = \frac{1}{8} \).
   Note, \( 0 \) just happens to also be an endpoint of \([0, 27]\).

(2) \( f(0) = \sqrt[3]{0^2 - \frac{4}{3} \cdot 0} = 0 \) \{ critical pts. \}

\[
\begin{align*}
   f\left( \frac{1}{8} \right) &= \sqrt[3]{\frac{1}{64} - \frac{4}{3} \cdot \frac{1}{8}} = \frac{1}{4} - \frac{1}{6} = \frac{3}{12} - \frac{2}{12} = \frac{1}{12} \\
   f\left( 27 \right) &= \sqrt[3]{27^2 - \frac{4}{3} \cdot 27} = 9 - 36 = -27 \\
\end{align*}
\]

\{ end pts. \}

(3) \( f\left( \frac{1}{8} \right) = \frac{1}{12} \) is abs. max.
(27) is abs. min.