Section 3.8 Derivatives of Inverse Functions and logarithms

Goals Find (and apply) formulas for:

(A) $\frac{d}{dx}\left[f^{-1}(x)\right]$

(B) $\frac{d}{dx}\left[\ln(x)\right]$

(C) $\frac{d}{dx}\left[\log_a(x)\right]$

(D) $\frac{d}{dx}\left[a^x\right]$

(A) What is $\frac{d}{dx}\left[f^{-1}(x)\right]$?

Know: $f\left(f^{-1}(x)\right) = x$ ← Differentiate both sides

$$\frac{d}{dx}\left[f\left(f^{-1}(x)\right)\right] = \frac{d}{dx}\left[x\right]$$

$$f'\left(f^{-1}(x)\right)\frac{d}{dx}\left[f^{-1}(x)\right] = 1$$ ← Chain rule

$$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'\left(f^{-1}(x)\right)}$$ ← Divide both sides by $f'\left(f^{-1}(x)\right)$

Thus: $$\boxed{\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'\left(f^{-1}(x)\right)}}$$ ← (Goal (A))

(B) Now we'll use this new rule to find $\frac{d}{dx}\left[\ln(x)\right]$.

Recall: If $f(x) = e^x$, then $f'(x) = e^x$ and $f^{-1}(x) = \ln(x)$.

$$\frac{d}{dx}\left[\ln(x)\right] = \frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{e^{f^{-1}(x)}} = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$$

Therefore

$$\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}$$

$$\frac{d}{dx}\left[\ln(g(x))\right] = \frac{1}{g(x)}g'(x)$$ ← Chain Rule applied to above ...

i.e. $$\frac{d}{dx}\left[\ln(g(x))\right] = \frac{g'(x)}{g(x)}$$ ← ... but its usually written this way

Text shows $\boxed{\frac{d}{dx}\left[\ln(|x|)\right] = \frac{1}{x}}$ ← will be useful in Chapter 5

Examples

$$\frac{d}{dx}\left[\ln(x^2+3x+1)\right] = \frac{2x+3}{x^2+3x+1}$$

$$\frac{d}{dx}\left[x\ln(x)\right] = (1)\ln x + x\frac{1}{x} = \ln(x)+1$$

$$\frac{d}{dx}\left[x^3\ln(x^2+\sqrt{x})\right] = 3x^2\ln(x^2+\sqrt{x}) + x^3\frac{2x+\frac{1}{2\sqrt{x}}}{x^2+\sqrt{x}}$$

$$\frac{d}{dx}\left[\ln(\sin(x)) + \frac{1}{x}\right] = \frac{\cos(x)}{\sin(x)} - \frac{1}{x^2} = \cot(x) - \frac{1}{x^2}$$

(C) What is $\frac{d}{dx}\left[\log_a(x)\right]$?

To answer this, remember the <u>change of base formula</u>:

$$\boxed{\log_a(x) = \frac{\ln(x)}{\ln(a)}}$$

Why it works: $\log_a(x) = \frac{\log_a(x)\ln(a)}{\ln(a)} = \frac{\ln\left(a^{\log_a(x)}\right)}{\ln(a)} = \frac{\ln(x)}{\ln(a)}$

Thus $\frac{d}{dx}\left[\log_a(x)\right] = \frac{d}{dx}\left[\frac{\ln(x)}{\ln(a)}\right] = \frac{1}{\ln(a)}\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{\ln(a)}\frac{1}{x}$

<u>New Formula</u>:

$$\boxed{\frac{d}{dx}\left[\log_a(x)\right] = \frac{1}{x\ln(a)}}$$

(D) What is $\frac{d}{dx}\left[a^x\right]$?

<u>Note</u>: $a^x = e^{\ln(a^x)} = e^{x\ln(a)}$

so... $\frac{d}{dx}\left[a^x\right] = \frac{d}{dx}\left[e^{x\ln(a)}\right] = e^{x\ln(a)}\cdot\frac{d}{dx}\left[x\ln(a)\right]$

$= e^{\ln(a^x)}\cdot\ln(a)$

$= a^x\ln(a)$

<u>New Formula</u>

$$\boxed{\frac{d}{dx}\left[a^x\right] = a^x\ln(a)}$$

In calculus, we tend to use e^x and $\ln(x)$ instead of a^x and $\log_a(x)$ so the previous two formulas don't come up very often.

We round out today's discussion with a significant technique.

LOGARITHMIC DIFFERENTIATION

Often the properties of ln can be used to differentiate functions that no rules apply to directly. This process is called logarithmic differentiation.

<u>Example</u> Find the derivative of $y = (x^2+1)^{x^3+x}$

This is a variable expression to a variable power – it is neither a power function nor an exponential function. To find its derivative, we first take ln of both sides and use log properties to simplify. Then we differentiate implicitly.

$$y = (x^2+1)^{x^3+x}$$

$$\ln(y) = \ln\left((x^2+1)^{x^3+x}\right)$$

$$\ln(y) = (x^3+x)\ln(x^2+1)$$

$$\frac{d}{dx}\left[\ln(y)\right] = \frac{d}{dx}\left[(x^3+x)\ln(x^2+1)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = (3x+1)\ln(x^2+1) + (x^3+x)\frac{2x}{x^2+1}$$

$$\frac{dy}{dx} = y\left((3x+1)\ln(x^2+1) + (x^3+x)\frac{2x}{x^2+1}\right)$$

$$\boxed{\frac{dy}{dx} = (x^2+1)^{x^3+x}\left((3x+1)\ln(x^2+1) + (x^3+x)\frac{2x}{x^2+1}\right)}$$

Read this section carefully. Check out how the text uses these ideas to show that the power rule $\frac{d}{dx}\left[x^n\right] = nx^{n-1}$ works, not just when n is an integer, but for <u>any real number n</u>.

Do lots of exercises. Master this material. It will be used often