Section 3.8 Derivatives of Inverse Functions and Logarithms

Goals Find (and apply) formulas for:

A \( \frac{d}{dx} \left[ f^{-1}(x) \right] \)

B \( \frac{d}{dx} \left[ \ln(x) \right] \)

C \( \frac{d}{dx} \left[ \log_a(x) \right] \)

D \( \frac{d}{dx} \left[ a^x \right] \)

(A) What is \( \frac{d}{dx} \left[ f^{-1}(x) \right] \)?

Know: \( f \left( f^{-1}(x) \right) = x \) \{differentiate both sides\}

\( \frac{d}{dx} \left[ f \left( f^{-1}(x) \right) \right] = \frac{d}{dx} [x] \)

\( f' \left( f^{-1}(x) \right) \frac{d}{dx} \left[ f^{-1}(x) \right] = 1 \) \{chain rule\}

\( \frac{d}{dx} \left[ f^{-1}(x) \right] = \frac{1}{f' \left( f^{-1}(x) \right)} \) \{divide both sides by \( f' \left( f^{-1}(x) \right) \}\)

Thus: \( \frac{d}{dx} \left[ f^{-1}(x) \right] = \frac{1}{f' \left( f^{-1}(x) \right)} \) \{Goal A\}

(B) Now we’ll use this new rule to find \( \frac{d}{dx} \left[ \ln(x) \right] \).

Recall: If \( f(x) = e^x \) then \( f'(x) = e^x \) and \( f^{-1}(x) = \ln(x) \)

\( \frac{d}{dx} \left[ \ln(x) \right] = \frac{d}{dx} \left[ f^{-1}(x) \right] = \frac{1}{f' \left( f^{-1}(x) \right)} = \frac{1}{e^{\ln(x)}} = \frac{1}{x} \)

Therefore

\[ \frac{d}{dx} \left[ \ln(x) \right] = \frac{1}{x} \]

\[ \frac{d}{dx} \left[ \ln(g(x)) \right] = \frac{1}{g(x)} \cdot g'(x) \]

i.e. \( \frac{d}{dx} \left[ \ln(g(x)) \right] = \frac{g'(x)}{g(x)} \)
Examples:
\[
\frac{d}{dx}[\ln(x^2+3x+1)] = \frac{2x+3}{x^2+3x+1}
\]
\[
\frac{d}{dx}[\ln(x)] = \ln'(x) = \frac{1}{x}
\]
\[
\frac{d}{dx}[\ln(x^2+\sqrt{x})] = \frac{2x}{x^2+\sqrt{x}}
\]
\[
\frac{d}{dx}[\ln(\sin(x)+\frac{1}{x})] = \frac{\cos(x)}{\sin(x)} - \frac{1}{x^2} = \cot(x) - \frac{1}{x^2}
\]

C) What is \(\frac{d}{dx}[\log_a(x)]\)?

To answer this, remember the change of base formula:
\[
\log_a(x) = \frac{\ln(x)}{\ln(a)}
\]

Why it works:
\[
\log_a(x) = \frac{\log_a(x) \ln(a)}{\ln(a)} = \frac{\ln(x)}{\ln(a)} = \frac{\ln(x)}{\ln(a)}
\]

Thus
\[
\frac{d}{dx}[\log_a(x)] = \frac{d}{dx}[\frac{\ln(x)}{\ln(a)}] = \frac{1}{\ln(a)} \frac{d}{dx}[\ln(x)] = \frac{1}{\ln(a)} \cdot \frac{1}{x}
\]

New Formula:
\[
\frac{d}{dx}[\log_a(x)] = \frac{1}{x \ln(a)}
\]

D) What is \(\frac{d}{dx}[a^x]\)?

Note: \(a^x = e^{\ln(a^x)} = e^{x \ln(a)}\)

So...
\[
\frac{d}{dx}[a^x] = \frac{d}{dx}[e^{x \ln(a)}] = e^{x \ln(a)} \cdot \frac{d}{dx}[x \ln(a)]
\]

New Formula:
\[
\frac{d}{dx}[a^x] = a^x \ln(a)
\]
In calculus, we tend to use \( e^x \) and \( \ln(x) \) instead of \( a^x \) and \( \log_a(x) \), so the previous two formulas don’t come up very often. We round out today’s discussion with a significant technique.

**Logarithmic Differentiation**

Often the properties of \( \ln \) can be used to differentiate functions that no rules apply to directly. This process is called logarithmic differentiation.

**Example** Find the derivative of \( y = (x^2 + 1)^{x^3+x} \).

This is a variable expression to a variable power — it is neither a power function nor an exponential function. To find its derivative, we first take \( \ln \) of both sides and use log properties to simplify. Then we differentiate implicitly.

\[
\begin{align*}
y &= (x^2 + 1)^{x^3+x} \\
\ln(y) &= \ln((x^2 + 1)^{x^3+x}) \\
\ln(y) &= (x^3+x) \ln(x^2+1)
\end{align*}
\]

\[
\frac{d}{dy} \left[ \ln(y) \right] = \frac{d}{dx} \left[ (x^3+x) \ln(x^2+1) \right]
\]

\[
\frac{1}{y} \frac{dy}{dx} = (3x+1) \ln(x^2+1) + (x^3+x) \frac{2x}{x^2+1}
\]

\[
\frac{dy}{dx} = y \left( (3x+1) \ln(x^2+1) + (x^3+x) \frac{2x}{x^2+1} \right)
\]

\[
\frac{dy}{dx} = \left( x^2 + 1 \right)^{x^3+x} \left( (3x+1) \ln(x^2+1) + (x^3+x) \frac{2x}{x^2+1} \right)
\]

Read this section carefully. Check out how the text uses these ideas to show that the power rule \( \frac{d}{dx} [x^n] = nx^{n-1} \) works, not just when \( n \) is an integer, but for any real number \( n \).

Do lots of exercises. Master this material. It will be used often.