Section 3.7  Implicit Differentiation

We can now differentiate just about any function \( y = f(x) \).

- \( y = f(x) = x^2 \)
  - \( y \) is an (explicit) function of \( x \) and we can find \( f'(x) \).

- But what about this situation?
  - Here \( y \) is related to \( x \) but it's not an explicit function of \( x \).
  - It's not so clear what the derivative would be.

- There are lots of functions \( y = f(x) \) satisfying \( x^2 + y^2 = 25 \).
- They are called implicit functions of the equation \( x^2 + y^2 = 25 \).

**Today's Goal:**
Discover how to differentiate implicit functions

**Motivational Problem**
Find slope of tangent to graph of \( x^2 + y^2 = 25 \) at point \( (3, -4) \)

Let \( y = f(x) \) be the above implicit function. The answer will be \( f'(3) \).

\[
\begin{align*}
  x^2 + y^2 &= 25 \\
  x^2 + (f(x))^2 &= 25 \\
  \frac{d}{dx} \left[ x^2 + (f(x))^2 \right] &= \frac{d}{dx} [25] \\
  2x + 2f(x)f'(x) &= 0 \\
  2f(x)f'(x) &= -2x \\
  f'(x) &= \frac{-2x}{2f(x)} = \frac{-x}{f(x)} \\
  f'(3) &= \frac{-2 \cdot 3}{2f(3)} = \frac{-3}{f(3)} = \frac{-3}{-4} = \frac{3}{4}
\end{align*}
\]

This process is called implicit differentiation.
Example

\[ x^2 + xy + y^2 = 7 \]

Find slope of tangent to the graph at (1,2)

Think this: \( y = f(x) \)

\[ x^2 + xy + y^2 = 7 \]

\[ x^2 + xf(x) + (f(x))^2 = 7 \]

\[ \frac{d}{dx} \left[ x^2 + xf(x) + (f(x))^2 \right] = \frac{d}{dx} [7] \]

\[ 2x + f(x) + xf'(x) + 2f(x)f'(x) = 0 \]

\[ xf'(x) + 2f(x)f'(x) = -2x - f(x) \]

\[ f'(x)(x + 2f(x)) = -2x - f(x) \]

\[ f'(x) = \frac{-2x - f(x)}{x + 2f(x)} \]

Slope = \[ f'(1) = \frac{-2(1) - f(1)}{1 + 2f(1)} = \frac{-2 - 2}{1 + 2 \cdot 2} = \frac{-4}{5} \]

Normal line at (1,2)

Tangent line at (1,2)

See text: Normal line at a point \( P \) is line perpendicular to tangent at \( P \).

\[ \frac{dy}{dx} \bigg|_{(x,y)=(1,2)} = \frac{-2 \cdot 1 - 2}{1 + 2 \cdot 2} = \frac{-4}{5} \]

Slope of normal line = \[ -\frac{1}{\text{slope of tangent}} = -\frac{-4}{5} = \frac{5}{4} \]
Ex
Find the slope of tangent to graph of $\sin xy = \cos xy$ at the point $(\pi, \frac{1}{4})$.

Note: $\sin \left( \pi \cdot \frac{1}{4} \right) = \frac{1}{\sqrt{2}} = \cos \left( \pi \cdot \frac{1}{4} \right)$ so $(\pi, \frac{1}{4})$ is on graph.

Solution:

$$\sin(xy) = \cos(xy)$$

$$D_x \left[ \sin(xy) \right] = D_x \left[ \cos(xy) \right]$$

$$\cos(xy)(y+xy') = -\sin(xy)(y+xy')$$

$$xy'\cos(xy) + y\cos(xy) = -xy'\sin(xy) - y\sin(xy)$$

$$xy'\cos(xy) + xy'\sin(xy) = -y\sin(xy) - y\cos(xy)$$

$$y'(x\cos(xy) + x\sin(xy)) = -y\sin(xy) - y\cos(xy)$$

$$y' = \frac{-y\sin(xy) - y\cos(xy)}{x\cos(xy) + x\sin(xy)} = \frac{-y}{x}$$

Slope at $(\pi, \frac{1}{4})$ is $-\frac{1}{\frac{1}{4}} = -y_{\pi} \approx -0.08$

Graph:

- $\sin xy = \cos xy$
- $\frac{\sin xy}{\cos xy} = 1$
- $\tan(xy) = 1$
- $xy = \frac{\pi}{4} + k\pi$
- $y = \left( \frac{\pi}{4} + k\pi \right) \frac{1}{x}$