Section 3.2 The Derivative as a Function

Recall:
The derivative of \( f(x) \) at point \( a \) is the number

\[
 f'(a) = \lim_{{h \to 0}} \frac{{f(a+h) - f(a)}}{h}
\]

= slope of tangent to \( y = f(x) \) at \( (a, f(a)) \)

Example
Last time we saw that if \( f(x) = \sqrt{x} \), then \( f'(a) = \frac{1}{2\sqrt{a}} \)

Notice how in this setup the number \( a \) can vary.
Since it is a variable, there is no harm in calling it \( x \).

Definition
The derivative of a function \( f(x) \) is another function denoted as \( f'(x) \) and defined as:

\[
 f'(x) = \lim_{{h \to 0}} \frac{{f(x+h) - f(x)}}{h}
\]

Moreover, \( f'(x) = \) slope of tangent to \( y = f(x) \) at \( (x, f(x)) \).

Always keep this picture in mind:

\[
y = f(x) \\
m = f'(x) = \lim_{{h \to 0}} \frac{{f(x+h) - f(x)}}{h}
\]
Example
The derivative of $f(x) = \sqrt{x}$ is the function $f'(x) = \frac{1}{2\sqrt{x}}$.
Example  Here's the graph of an $f(x)$. Sketch graph of $f'(x)$

\[ y = f(x) \]
\[ y = f'(x) \]

<table>
<thead>
<tr>
<th>$x$</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f'(x)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Slope of $y = f(x)$ at $(x, f(x))$.

Exercise  For $f(x) = \sqrt[3]{x}$, show $f'(x) = \frac{1}{3\sqrt[3]{x^2}}$

\[ y = f(x) \]
\[ y = f'(x) \]

Note  Even though $f(0) = \sqrt[3]{0} = 0$ is defined, $f'(0) = \frac{1}{3\sqrt[3]{0^2}} = \frac{1}{0}$ is not defined. Reason: Tangent at $(0, f(0) = (0, 0)$ is vertical. Its slope is not defined.

Notation  If $y = f(x)$ then

\[ f'(x) = \frac{dy}{dx} = y' = \frac{d}{dx} f(x) = \frac{d}{dx} [f(x)] = D_x f(x) \]

\[ f'(a) = \frac{dy}{dx} \big|_{x=a} = y' \big|_{x=a} \]

Read the text and the examples therein. There are a number of enlightening examples that we have not discussed here. I try not to duplicate text examples.