Section 3.2 The Derivative as a Function

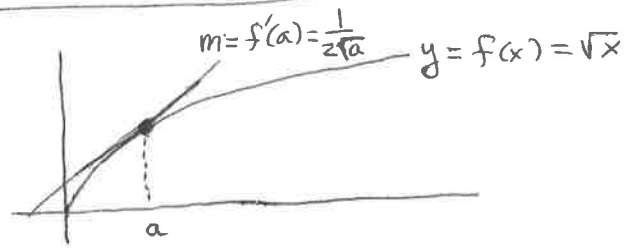
Recall:

The derivative of $f(x)$ at point a is the number

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

$$= \text{slope of tangent to } y = f(x) \text{ at } (a, f(a))$$

Example Last time we saw that if $f(x) = \sqrt{x}$, then $f'(a) = \frac{1}{2\sqrt{a}}$.



Notice how in this set up the number a can vary. Since it is a variable, there is no harm in calling it x.

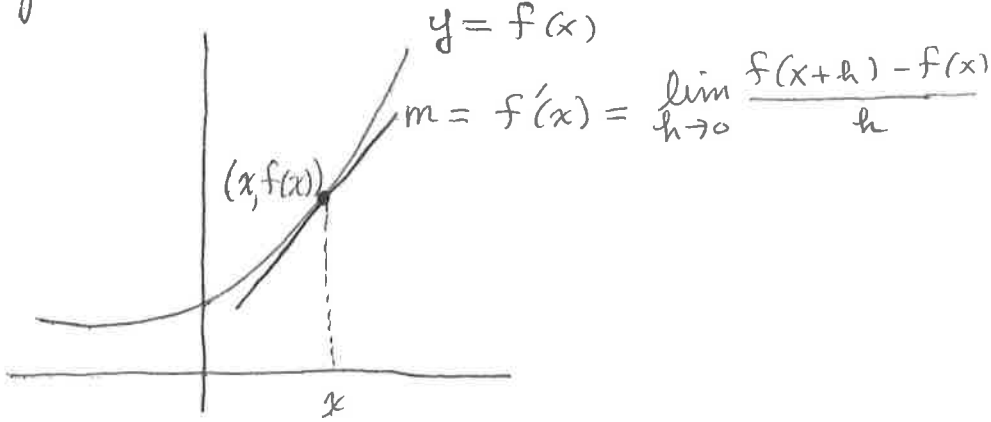
☆☆ **Definition**

The derivative of a function $f(x)$ is another function denoted as $f'(x)$ and defined as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

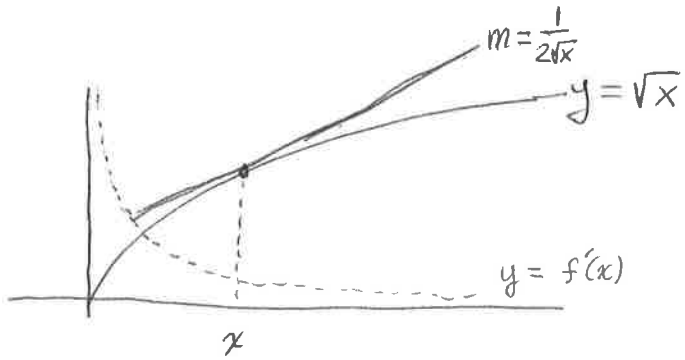
Moreover, $f'(x) =$ slope of tangent to $y = f(x)$ at $(x, f(x))$.

Always keep this picture in mind:



Example

The derivative of $f(x) = \sqrt{x}$ is the function $f'(x) = \dfrac{1}{2\sqrt{x}}$



Example

Find the derivative of $f(x) = x^2 + 2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

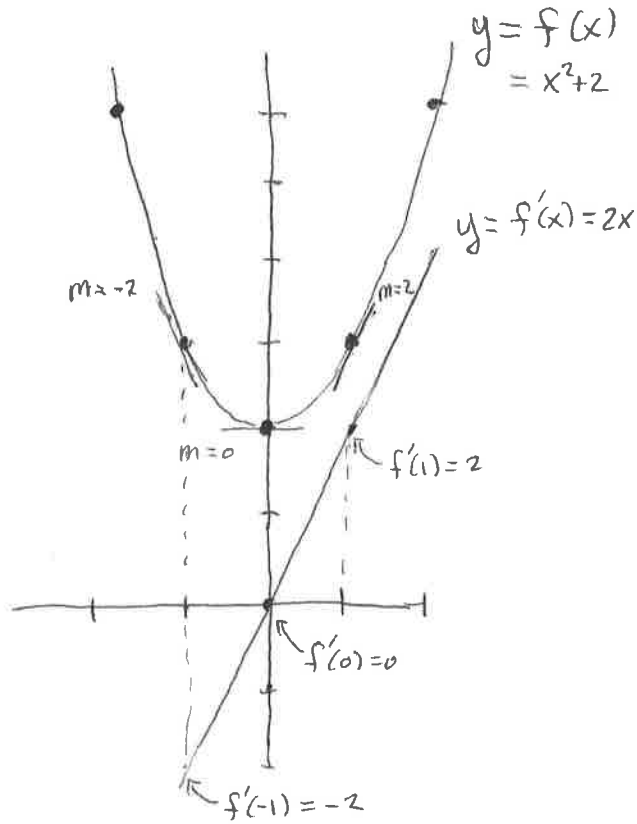
$$= \lim_{h \to 0} \frac{(x+h)^2 + 2 - (x^2+2)}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 2 - x^2 - 2}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} \frac{h(2x+h)}{h}$$

$$= \lim_{h \to 0} 2x + h = \boxed{2x}$$

Therefore $\boxed{f'(x) = 2x}$



Example Find The derivative of $f(x) = \frac{1}{x}$

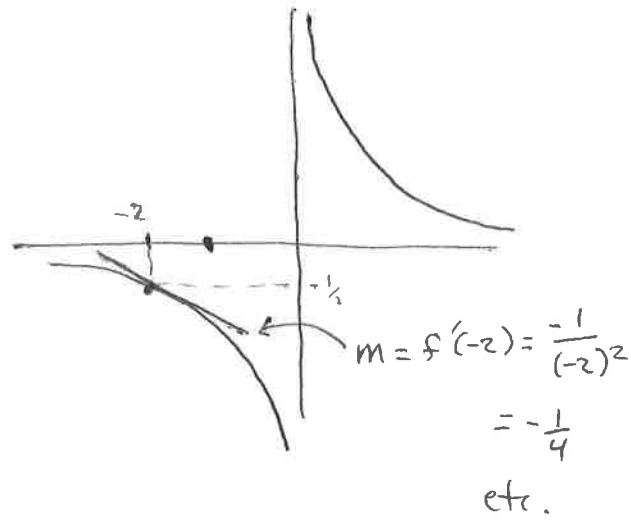
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

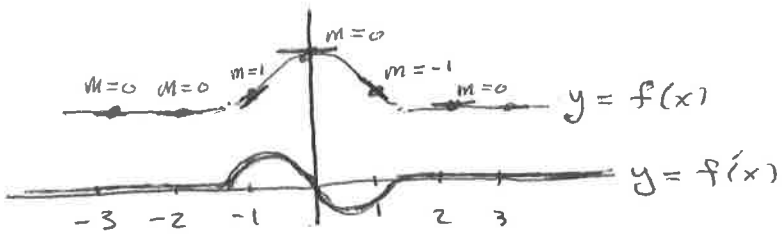
$$= \lim_{h \to 0} \frac{\frac{x - (x+h)}{(x+h)x}}{\frac{h}{1}}$$

$$= \lim_{h \to 0} \frac{-h}{(x+h)x} \cdot \frac{1}{h} = \lim_{h \to 0} \frac{-1}{(x+h)x} = \frac{-1}{x^2}$$

Therefore $\boxed{f'(x) = -\frac{1}{x^2}}$



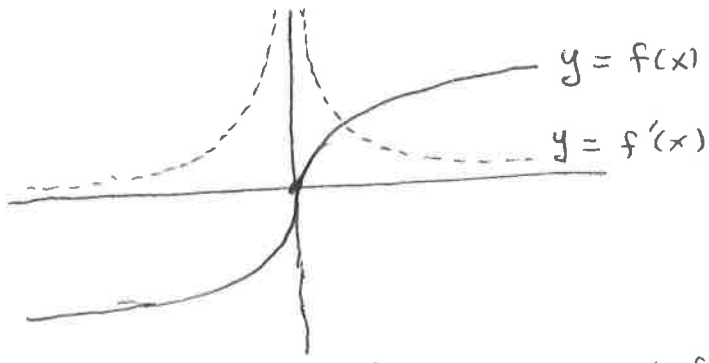
Example Here's the graph of an $f(x)$. Sketch graph of $f'(x)$



x	-3	-2	-1	0	1	2	3
$f'(x)$	0	0	1	0	-1	0	0

(slope of $y=f(x)$ at $(x, f(x))$)

Exercise For $f(x) = \sqrt[3]{x}$, show $f'(x) = \dfrac{1}{3\sqrt[3]{x}^{\,2}}$



Note Even though $f(0) = \sqrt[3]{0} = 0$ is defined, $f'(0) = \dfrac{1}{3\sqrt[3]{0}^{\,2}} = \dfrac{1}{0}$ is not defined. Reason: Tangent at $(0, f(0)) = (0,0)$ is vertical. Its slope is not defined.

NOTATION If $y = f(x)$ then

$$f'(x) = \frac{dy}{dx} = y' = \frac{d}{dx} f(x) = \frac{d}{dx}[f(x)] = D_x f(x)$$

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left. y' \right|_{x=a}$$

Read the text and the examples therein. There are a number of enlightening examples that we have not discussed here. Lecture text I try not to duplicate text examples.