Section 2.6 Limit Involving Infinity (Continued)

Last time: Making sense of \( \lim_{x \to \infty} f(x) = L \)

Today: Making sense of \( \lim_{x \to \infty} f(x) = \infty \)

Two simple examples serve as guides

\[
\begin{align*}
\lim_{x \to 0} \frac{1}{x} &= \text{DNG} & f(x) &= \frac{1}{x} \\
\lim_{x \to 0^+} \frac{1}{x} &= \infty \\
\lim_{x \to 0^-} \frac{1}{x} &= -\infty
\end{align*}
\]

\[
\begin{align*}
\lim_{x \to 0} \frac{1}{x^2} &= \text{DNG} \\
\lim_{x \to 0^+} \frac{1}{x^2} &= \infty \\
\lim_{x \to 0^-} \frac{1}{x^2} &= \infty
\end{align*}
\]

\[\lim_{x \to c} f(x) = \infty \quad \text{if, no matter how big you want } f(x) \text{ to be, it can be made that big by taking } x \text{ sufficiently close to } c \quad \text{(and to right of } c)\]

\[\text{Example} \quad f(x) = \frac{x-1}{x^2-1} = \frac{x-1}{(x-1)(x+1)} = \frac{1}{x+1}\]

- \( \lim_{x \to 1} \frac{x-1}{x^2-1} = \lim_{x \to 1} \frac{1}{x+1} = \frac{1}{2} \)
- \( \lim_{x \to 1^+} \frac{x-1}{x^2-1} = \infty \)
- \( \lim_{x \to 1^-} \frac{x-1}{x^2-1} = -\infty \)

**Definition:** If \( \lim_{x \to c^+} f(x) = \pm \infty \) or \( \lim_{x \to c^-} f(x) = \pm \infty \), then the line \( x = c \) is called a vertical asymptote of \( f(x) \).
Ex \[ \csc(x) = \frac{1}{\sin(x)} \] has vertical asymptotes \[ x = k\pi \text{ for } k = 0, \pm 1, \pm 2, \ldots \]

Typically, the vertical asymptotes of \( y = f(x) \) will be lines \( x = c \) for which \( f(c) \) has a zero in the denominator. But you must work out \( \lim_{x \to c^\pm} f(x) \) to know for sure.

Ex Find vertical and horizontal asymptotes of
\[ f(x) = \frac{x^3 - x^2 - 6x}{x^3 + x^2 - 2x} \]

Horizontal Asymptotes:
\[ \lim_{x \to \infty} f(x) = 1 = \lim_{x \to -\infty} f(x) \]
Thus line \( y = 1 \) is H.A.

Vertical asymptotes:
\[ f(x) = \frac{x^3 - x^2 - 6x}{x^3 + x^2 - 2x} = \frac{x(x^2 - x - 6)}{x(x^2 + x - 2)} = \frac{x(x-3)(x+2)}{x(x+2)(x-1)} = \frac{x-3}{x-1} \]

Check
\[ \lim_{x \to 0} f(x) = \frac{0-3}{0-1} = 3 \quad \text{Not } \infty \quad \text{No V.A.} \]
\[ \lim_{x \to -2} f(x) = \frac{-2-3}{-2-1} = \frac{5}{3} \]
\[ \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{x-3}{x-1} = -\infty \]
\[ \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{x-3}{x-1} = \infty \]
\( \{ \begin{array}{l} \text{denominator is zero at } x = 0, x = -2, x = 1 \\ \text{these are candidates for vertical asymptote} \end{array} \)
**Examples**

\[ \lim_{x \to \pi^+} \frac{x}{\cos(x) + 1} = \infty \]  
(Applies, \( n \), positive)

\[ \lim_{x \to \pi^-} \frac{x}{\cos(x) + 1} = \infty \]  
(Applies, \( n \), positive)

Thus \( x = \pi \) is a **V.A.** of \( f(x) = \frac{x}{\cos(x) + 1} \).

Are there other vertical asymptotes? **YES** at any \( x \) that makes \( \cos(x) = -1 \), so denominator is zero. Thus \( x = \pi + 2k\pi = (1 + 2k)\pi \).

**VA at** \( x = n\pi \) **for any odd** \( n \)

**Oblique Asymptotes**

Consider \( f(x) = x + \frac{1}{x} \)

Line \( y = 0 \) is **VA**.

Line \( y = x \) is **oblique asymptote** (neither vertical nor horizontal)