Section 2.6 Limits Involving Infinity (Continued)

Last time: Making sense of $\lim_{x\to\infty} f(x) = L$

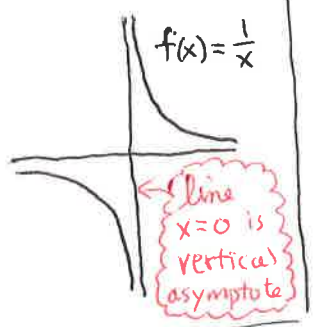
Today: Making sense of $\lim_{x\to c} f(x) = \infty$

Two simple examples serve as guides

$\lim_{x\to 0} \frac{1}{x}$ DNE

$\lim_{x\to 0^+} \frac{1}{x} = \infty$

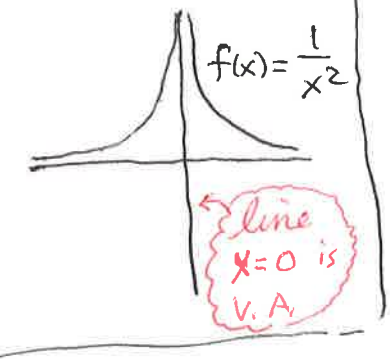
$\lim_{x\to 0^-} \frac{1}{x} = -\infty$



$\lim_{x\to 0} \frac{1}{x^2} = \infty$

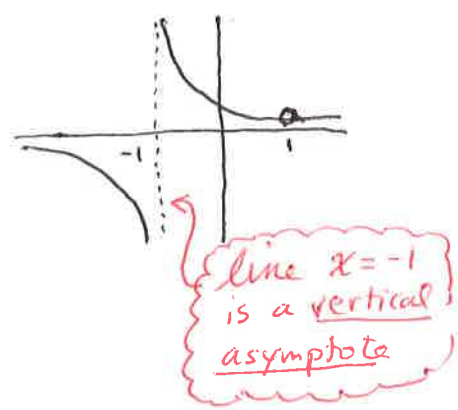
$\lim_{x\to 0^+} \frac{1}{x^2} = \infty$

$\lim_{x\to 0^-} \frac{1}{x^2} = \infty$



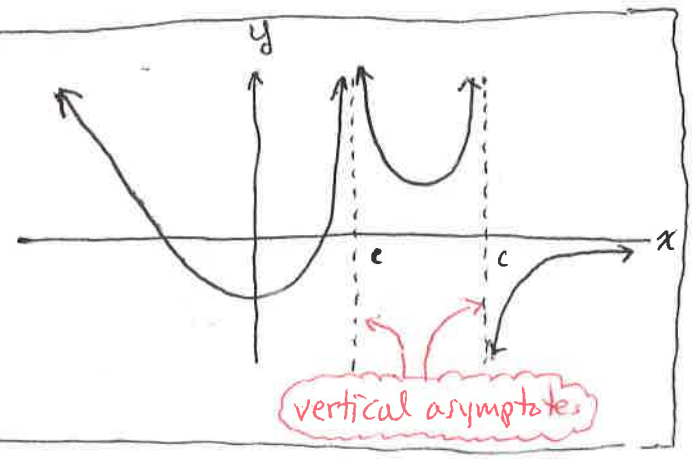
$\lim_{x\to c^+} f(x) = \infty$ if, no matter how big you want $f(x)$ to be, it can be made that big by taking x sufficiently close to c (and to right of c). etc.

Example $f(x) = \frac{x-1}{x^2-1} = \frac{x-1}{(x-1)(x+1)} = \frac{1}{x+1}$

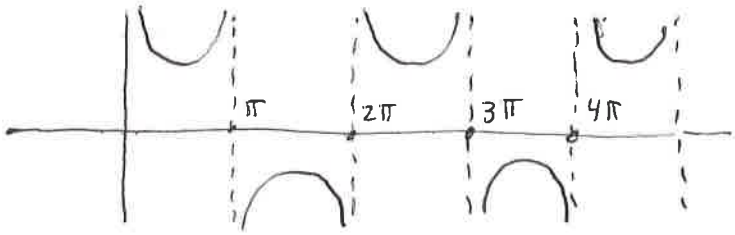


- $\lim_{x\to 1} \frac{x-1}{x^2-1} = \lim_{x\to 1} \frac{1}{x+1} = \boxed{\frac{1}{2}}$
- $\lim_{x\to -1^+} \frac{x-1}{x^2-1} = \infty$
- $\lim_{x\to -1^-} \frac{x-1}{x^2-1} = -\infty$

Definition If $\lim_{x\to c^+} f(x) = \pm\infty$ or $\lim_{x\to c^-} f(x) = \pm\infty$, then the line $x=c$ is called a <u>vertical asymptote</u> of $f(x)$



Ex $\csc(x) = \dfrac{1}{\sin(x)}$ has vertical asymptotes $x = k\pi$ for $k = 0, \pm 1, \pm 2, \ldots$



> Typically, the vertical asymptotes of $y = f(x)$ will be lines $x = c$ for which $f(c)$ has a zero in the denominator.
> **But** you must work out $\lim_{x \to c^{\pm}} f(x)$ to know for sure.

Ex Find vertical and horizontal asymptotes of

$$f(x) = \frac{x^3 - x^2 - 6x}{x^3 + x^2 - 2x}.$$

Horizontal Assymptotes:

$$\lim_{x \to \infty} f(x) = 1 = \lim_{x \to -\infty} f(x)$$

$\boxed{\text{Thus line } y = 1 \text{ is H.A.}}$

Vertical asymptotes:

$$f(x) = \frac{x^3 - x^2 - 6x}{x^3 + x^2 - 2x} = \frac{x(x^2 - x - 6)}{x(x^2 + x - 2)} = \frac{x(x-3)(x+2)}{x(x+2)(x-1)} = \frac{x-3}{x-1}$$

denominator is zero at $x = 0$ $x = -2$, $x = 1$ these are candidates for vertical asymptotes

Check

$$\lim_{x \to 0} f(x) = \frac{0-3}{0-1} = \boxed{3}$$

$$\lim_{x \to -2} f(x) = \frac{-2-3}{-2-1} = \boxed{\frac{5}{3}}$$

Not ∞ No V.A.

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{x-3}{x-1} = -\infty$$

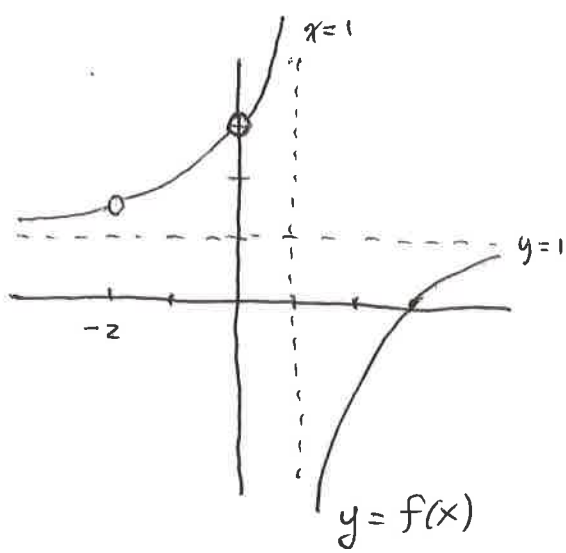
approaches -2

approaches 0, positive

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{x-3}{x-1} = \infty$$

approaches 0, negative

$x = 1$ is V.A



Examples

Approaches π, positive

$$\lim_{x \to \pi^+} \frac{x}{\cos(x)+1} = \infty$$

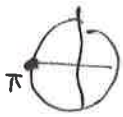
Approaches 0, positive

$$\lim_{x \to \pi^-} \frac{x}{\cos(x)+1} = \infty$$

$$\lim_{x \to \pi} \frac{x}{\cos(x)+1} = \infty$$

Thus $x = \pi$ is a V.A of $f(x) = \dfrac{x}{\cos(x)+1}$.

Are there other vertical asymptotes. <u>YES</u> at any x that makes $\cos(x) = -1$ so denominator is zero.



Thus $x = \pi + 2k\pi = (1+2k)\pi$.

$$\boxed{\text{VA at } x = n\pi \text{ for any } \underline{\text{odd}}\ n}$$

Oblique Asymptotes

Consider $f(x) = x + \dfrac{1}{x}$

line $y = 0$ is VA.

line $y = x$ is <u>oblique</u> <u>asymptote</u>

(neither vertical nor horizontal)