Section 4.3

(34) Graph $g(x) = e^{-\frac{x^2}{2}}$

First examine where $g(x)$ is increasing / decreasing:

$$g'(x) = e^{-\frac{x^2}{2}}(-x) = -x\,e^{-\frac{x^2}{2}}$$

	0		
+ + + + + +		+ + + + +	$e^{-\frac{x^2}{2}}$
+ + + + + +		− − − − −	$-2x$
+ + + + + +		− − − − −	$g'(x) = -x e^{-\frac{x^2}{2}}$

Conclusion: $g(x)$ increasing on $(-\infty, 0)$
$g(x)$ decreasing on $(0, \infty)$

Now let's examine the concavity.

$$g''(x) = -e^{-\frac{x^2}{2}} - x e^{-\frac{x^2}{2}}(-x) =$$

$$= e^{-\frac{x^2}{2}}(-1 + x^2)$$

$$= e^{-\frac{x^2}{2}}(x-1)(x+1)$$ ← Note: $g''(x) = 0$ for $x = \pm 1$

	−1		0		1		
+ + +	o	− − −		− − −	o	+ + +	$g''(x)$

Conclusion: $g(x)$ concave up on $(-\infty, -1)$ and $(1, \infty)$
$g(x)$ concave down on $(-1, 1)$

Inflection Points: $(1, g(1)) = (1, e^{-1/2}) = (1, \frac{1}{\sqrt{e}}) \approx (1, 0.6)$
$(-1, g(-1)) = (-1, e^{-1/2}) = (-1, \frac{1}{\sqrt{e}}) \approx (-1, 0.6)$

We will use all this to sketch the graph, but first lets find its y-intercept so we can hang the graph there:

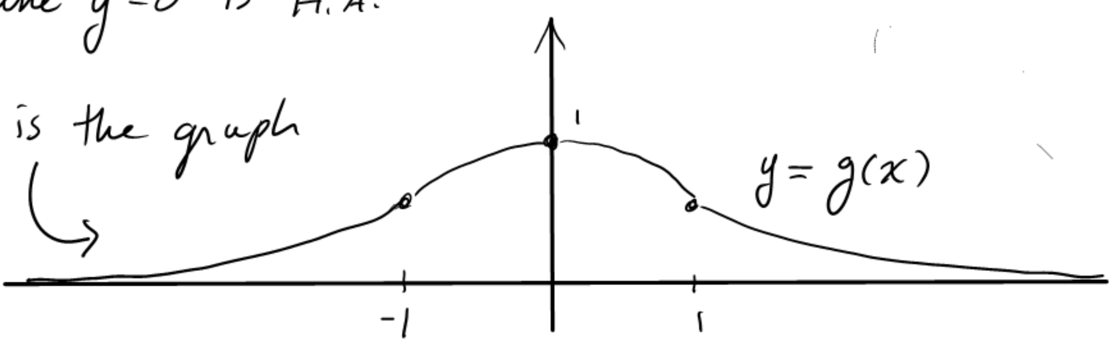
y-intercept $g(0) = e^{-0^2/2} = e^0 = 1$

Finally note that $\lim\limits_{x\to\infty} g(x) = \lim\limits_{x\to\infty} e^{-\frac{x^2}{2}} = 0$

and $\lim\limits_{x\to-\infty} g(x) = \lim\limits_{x\to-\infty} e^{-\frac{x^2}{2}} = 0$

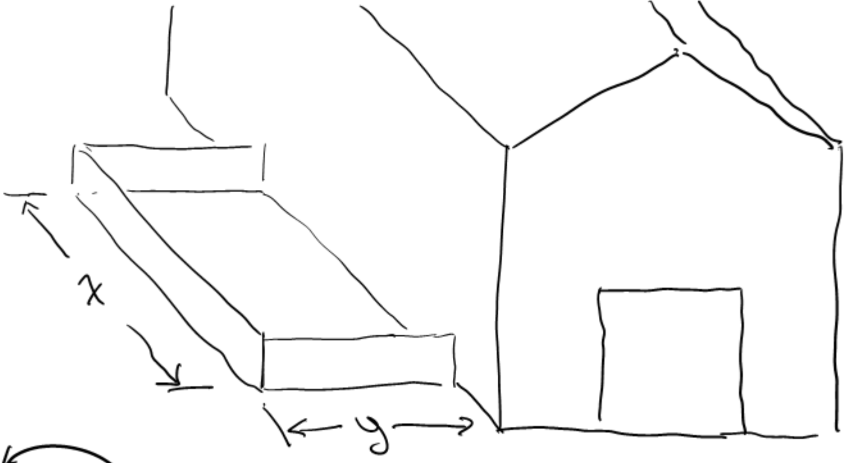
So line $y = 0$ is H.A.

Here is the graph



Section 4.4

(14) (a) 200 meters of fencing are available for this pen



What dimensions x, y give maximum area?

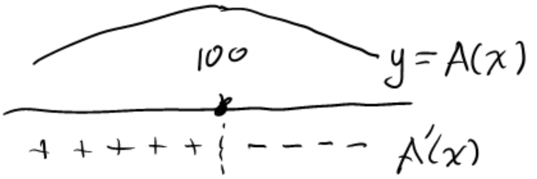
$$\text{Area} = xy = x\left(100 - \frac{x}{2}\right)$$
$$= 100x - \frac{x^2}{2}$$

Total fencing:
$200 = x + y + y$
so $2y = 200 - x$
$y = 100 - \frac{x}{2}$

$$\text{Area} = A(x) = 100x - \frac{x^2}{2}$$

Find absolute maximum of this

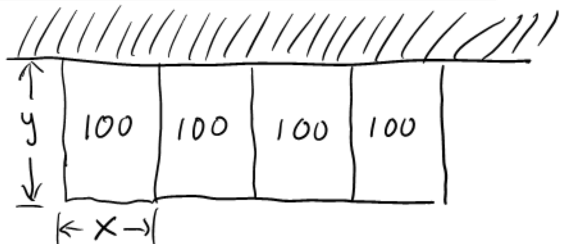
$A'(x) = 100 - x = 0$ critical point
$x = 100$



We get an <u>absolute</u> <u>maximum</u> of area when $x = 100$, so $y = 100 - \frac{100}{2}$

<u>Answer:</u> Put $x = 100$, $y = 50$ for maximum area

(b) Each pen has an area of 100 square meters. What dimension x, y of each pen minimize total amount of fence?



Total fence: $4x + 5y = 4x + 5\frac{100}{x}$ $xy = 100$, so $y = \frac{100}{x}$

$= L(x) = 4x + \frac{500}{x}$ ← Find absolute minimum of this

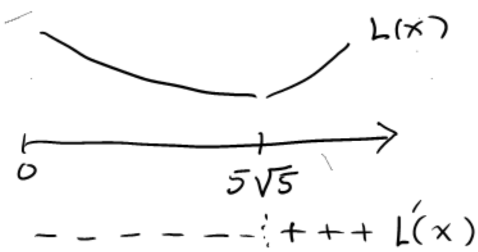
$L'(x) = 4 - \frac{500}{x^2} = 0$

$4 = \frac{500}{x^2}$

$4x^2 = 500$

$x^2 = 125$

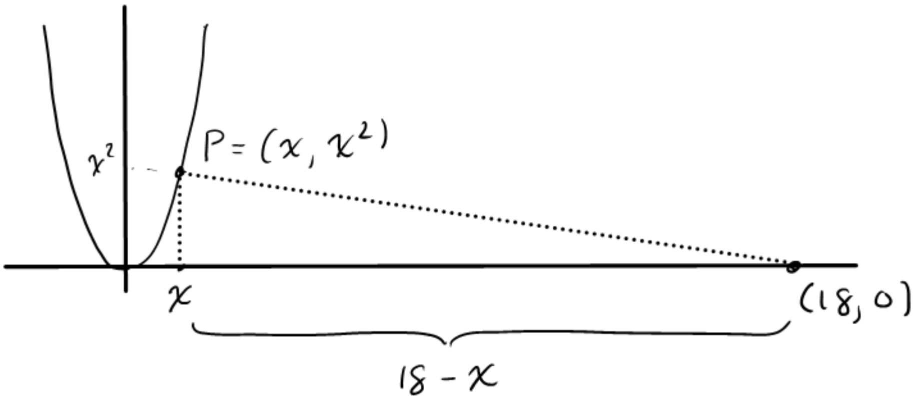
$x = \sqrt{125} = 5\sqrt{5}$ ← critical point



Get <u>abs</u> <u>min</u> at $x = 5\sqrt{5}$

<u>Answer:</u> To minimize fence, let $x = 5\sqrt{5}$, $y = \frac{100}{5\sqrt{5}} = \frac{20}{\sqrt{5}} = 4\sqrt{5}$

§4.4 20 Find the point P on $y = x^2$ that is closest to the point $(18, 0)$



Let the point P be (x, x^2)
By the distance formula, its distance from $(18,0)$ is

$$\sqrt{(18-x)^2 + (x^2)^2} = \sqrt{(18-x)^2 + x^4}$$

We seek the value of x that minimizes this. For this purpose, we can simplify things by squaring the distance and finding the x that minimizes that

Thus we want to find the absolute minimum of

$$D(x) = (18-x)^2 + x^4$$
$$D'(x) = 2(18-x)(-1) + 4x^3$$
$$= -36 + 2x + 4x^3$$
$$= 4x^3 + 2x - 36 = 0$$

$$2x^3 + x - 18 = 0$$
$$(x-2)(2x^2+4x+9) = 0$$

$x = 2$

Critical point: $x = 2$

Absolute minimum of $D(x)$ when $x=2$

2

$- - - - - - \mid + + + + + + + + + \quad D'(x) = 4x^3 + 2x - 36$

Answer: $P = (2, 2^2) = (2, 4)$ is closest to $(18,0)$.
Minimum distance is

$$\sqrt{(18-2)^2 + 4^2} = \sqrt{16^2 + 16} = \sqrt{16(16+1)} = 4\sqrt{17}$$