CHAPTER 2

Functions

CALCULUS is phrased in functions. To understand calculus you must first understand functions. This chapter is a brief refresher to ensure that your grip on functions is firm enough for the journey.

2.1 Functions and Their Graphs

For the purpose of this course, a **function** is a rule that describes a relationship between two variable quantities, say x and y, in such a way that y depends on x. One way of expressing a function is with an algebraic equation, such as

$$y = \sqrt{x} + 1.$$

This rule asserts that y equals the square root of whatever x is, plus 1. The variable x is called the **independent variable** or **input** of the function; y is called the **dependent variable** or **output**, because y *depends* on the value of x, which can be chosen *independently*.

We often use a letter such as f to denote a function, writing the above as

$$y = f(x),$$

where $f(x) = \sqrt{x} + 1$. That is, we let $f(x)$ stand for the expression $\sqrt{x} + 1$. In doing this we are using the letter f as the *name of the function*, and $f(x)$ stands for the output value resulting from an input value of x. For example, $f(4) = \sqrt{4} + 1 = 3$, meaning f returns output 3 when the input is 4.

name of function   independent variable, or *input*

$$\underbrace{f(x)}_{\text{output}}$$

In referring to such a function we may write either f or $f(x)$. The expression $f(x)$ reminds us that f returns a value $f(x)$ that depends on x.

Continuing with $f(x) = \sqrt{x} + 1$, note $f(0) = \sqrt{0} + 1 = 1$ and $f(1) = \sqrt{1} + 1 = 2$. Of course the input values need not only be integers:

$$f(0.25) = f\left(\frac{1}{4}\right) = \sqrt{\frac{1}{4}} + 1 = \frac{\sqrt{1}}{\sqrt{4}} + 1 = \frac{1}{2} + 1 = \frac{3}{2} = 1.5,$$

so that for an input of 0.25 we get an output of 1.5.

So far we have plugged in "convenient" input values x that are perfect squares, so that the square root is easy to compute. For other values a calculator may be helpful, as in $f(2) = \sqrt{2}+1 \approx 2.141$, where we have rounded off the output to three decimal places. Also $f(\pi) = \sqrt{\pi} + 1$. Although we could use a calculator to get a decimal approximation, it is perfectly fine to leave an answer as $\sqrt{2} + 1$ or $\sqrt{\pi} + 1$, which are, after all, exact values rather than approximations.

Observe that $f(-4) = \sqrt{-4} + 1$ cannot be computed (or at least is not a real number) because $\sqrt{-4}$ is not defined. This highlights the fact that some numbers may not be valid inputs for certain functions. We will have more to say about in the next section. For now, notice that $f(x) = \sqrt{x} + 1$ accepts only non-negative values for its input.

We also use the letters g and h to denote functions, so, for example, we $g(x) = x^2 + 1$ and $h(x) = 3x - 7$ are two functions. Of course we can also use other letters for the variables, for instance

$$h(w) = \frac{1+w}{w}$$

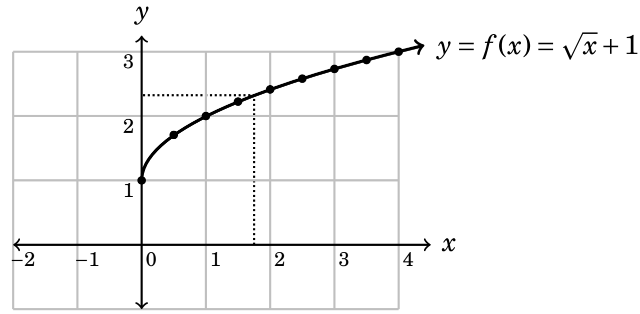
is a function whose independent variable is w.

A function's input can be a number or an expression representing a number. For example, with $f(x) = \sqrt{x} + 1$ we might encounter $f(x^2 + x - 3) = \sqrt{x^2 + x - 3} + 1$. For $h(w) = \dfrac{1+w}{w}$, notice that $h(x-1) = \dfrac{1+(x-1)}{x-1} = \dfrac{x}{x-1}$.

Any function can be represented or described visually by its *graph*. The **graph** of a function $f(x)$ is the set of all points $(x, f(x))$ on the plane, for all possible inputs x. For example, to draw a graph of $f(x) = \sqrt{x} + 1$ we might first create a table that pairs some sample input values x with their corresponding output values $f(x)$.

x	0	0.5	1	1.5	2	2.5	3	3.5	4
$f(x)$	0	0.717	1.000	2.224	2.414	2.581	2.732	2.871	3.000

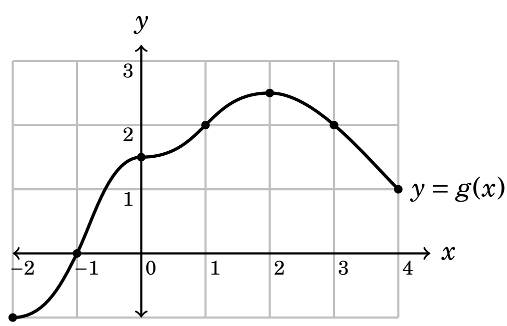
Plotting these points and connecting them with a smooth curve gives a picture of the graph, below.



A function's values can be estimated from its graph. Say we want to find $f(1.75)$. We draw a vertical line (dotted, in the picture above) from $x = 1.75$ to the graph. Then we move horizontally over to the y-axis, to arrive at what appears to be about $y = 2.3$. Thus $f(1.75) \approx 2.3$. (A calculator gives a slightly more accurate value of $y = \sqrt{1.75} + 1 \approx 2.3228$.)

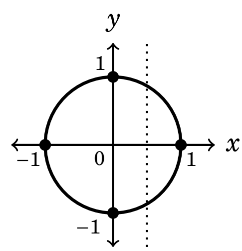
It is important to appreciate that a function's graph and its algebraic expression (such as $f(x) = \sqrt{x} + 1$) are two different ways of defining the function. The graph is the set of all points $(x, f(x))$, and this pairs each input x with its output $f(x)$. Therefore the graph encodes all information about $f(x)$ in a visual language. (Although in practice we tend to draw only part of the graph. The above graph does not include, say, the point $(9, f(9)) = (9, 4)$, so we can't read off $f(9) = 4$ from it.) In calculus, we often switch between the algebraic formulation of a function and its graph. They are two different ways of thinking about the function, and each one highlights features of the function that may be hidden in the other.

Our next example underscores the fact that a function can be defined entirely in terms of its graph. Consider the function $g(x)$ graphed below. Although this function is not defined by an algebraic expression, we can still obtain various function values: $g(-2) = -1$, $g(-1) = 0$, $g(0) = 1.5$, $g(3) = 2$, and $g(-0.5) = 1$, etc.



Continuing the example, if asked what x values make $g(x) = 2$, we see that $x = 1$ and $x = 3$ are the only two such values.

An important feature of a function is that each input x yields *exactly one* output $f(x)$. As a consequence, not every graph is the graph of a function. Consider the graph of the *equation* $x^2 + y^2 = 1$, which is a circle of radius 1 centered at the origin. If this were the graph of a function, we would be able to determine, say, $f(0.5)$ by drawing a vertical line through the point 0.5 on the x-axis and finding the y coordinate of the point at which it crosses the graph.

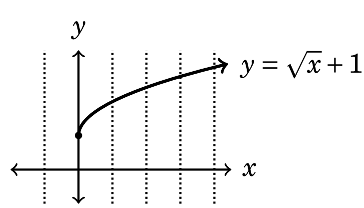


But the vertical line through $x = 0.5$ crosses the graph at *two* points, so it is impossible to say what $f(0.5)$ would be. For this reason, a vertical line that crosses a graph at more than one point signifies that the graph cannot be the graph of a function. This is called the *vertical line test*.

Vertical Line Test: A graph is the graph of a function if and only if no vertical line crosses the graph at more than one point.

Thus the graph of $x^2 + y^2 = 1$, above, is not the graph of a function because it fails the vertical line test. There is a vertical line that crosses the graph at more than one point. (In fact, there are many such lines.)

By contrast, the graph of $y = \sqrt{x} + 1$ passes the vertical line test. Any vertical line crosses the graph at exactly one point or at *no point at all*. But no vertical line crosses more than once. This *is* the graph of a function.



In studying calculus, in *thinking about calculus*, you will frequently find yourself sketching graphs to clarify what you are reading, or to analyze a problem. You will quickly internalize the vertical line test and use it subconsciously whenever you sketch a function's graph.

The **domain** of a function $f(x)$ is the set of all x-values for which $f(x)$ is defined. For example, consider the function $f(x) = \sqrt{x}+1$, whose graph is sketched on the right of Figure 2.1, below. We can't compute $\sqrt{x}$ for negative values of x, so $f(x)$ is only defined for $x \geq 0$. Thus the domain of $f(x)$ is the right half of the x-axis, including 0. The domain consists of all those x values with the property that a vertical line through x would cross the graph at exactly one point, at height $f(x)$. If x is *not* in the domain (in this case, if x is negative), then a vertical line through x would not cross the graph at all (because $f(x)$ is not defined).

The idea of a function's graph is so fundamental that we are going to restate it in somewhat more precise language: The **graph** of a function $f(x)$ is the set of all points $(x, f(x))$ on the plane, for which x is in the function's domain. Figure 2.1 illustrates this for two functions.

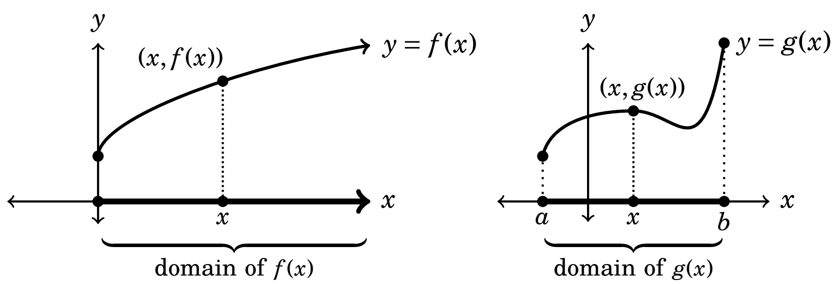


Figure 2.1. The **domain** of a function $f(x)$ is the set of all values x for which $f(x)$ is defined. The **graph** of $f(x)$ is the set of all points $(x, f(x))$, where x is in the domain of the function.

There is a slight difference between the *graph* of a function and a *drawing of the graph of a function*. The graph of $f(x)$ is the set of all points $(x, f(x))$ on the plane, where x is in the domain; this is a precise mathematical object and can be infinitely big, as in $f(x) = \sqrt{x}+1$, whose domain is all values of x from 0 to ∞. By contrast, a *drawing* of the graph (like the one in Figure 2.1) is necessary finite and may show only part of the graph. Further, in the drawing the curve has a physical thickness (to make it visible) and may be decorated with bolded points; these are features of the drawing, not of the theoretical graph.

We will blur this distinction. Wğe will refer to graphs and their drawings as essentially the same thing, even though this is a bit like considering a friend to be the same thing as a photograph of him.

Try a few of the following exercises. If you can do them quickly you are ready to move to the next section.

Exercises for Section 2.1

The functions in Exercises 1–6 will be featured in Section 2.3. Sketching their graphs now is good preparation.

1. On the same coordinate axis, draw the graphs of $f(x) = x^2$ and $g(x) = x^4$ by plotting points for $x = -1.5, -1, -0.5, 0, 0.5, 1, 1.5$.

2. On the same coordinate axis, draw the graphs of $f(x) = x^3$ and $g(x) = x^5$ by plotting points for $x = -1.5, -1, -0.5, 0, 0.5, 1, 1.5$.

3. On the same coordinate axis, draw the graphs of $f(x) = \dfrac{1}{x^2}$ and $g(x) = \dfrac{1}{x^4}$ by plotting points for $x = -1.5, -1, -0.5, 0, 0.5, 1, 1.5$.

4. On the same coordinate axis, draw the graphs of $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{1}{x^3}$ by plotting points for $x = -1.5, -1, -0.5, 0, 0.5, 1, 1.5$.

5. Sketch the graph of $f(x) = \sqrt[3]{x}$ for $-8 \le x \le 8$.

6. Sketch the graph of $f(x) = |x|$ for $-4 \le x \le 4$.

7. For $f(x) = x + \dfrac{1}{x}$, find $f\left(\dfrac{1}{x+1}\right)$.

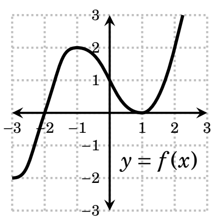
8. For $f(x) = x + \dfrac{1}{x}$, find $f\left(x + \dfrac{1}{x}\right)$.

9. For $f(x) = \sqrt{x+1}$, find $f(x^2 - 2x)$.

10. For $f(x) = \sqrt{x+1}$, find $f(x^2 + 3)$.

11. Answer the questions about $f(x)$.

a. $f(-3) =$
b. $f(0) =$
c. $f(2) =$
d. Solve: $f(x) = 1$
e. Solve: $f(x) = x$



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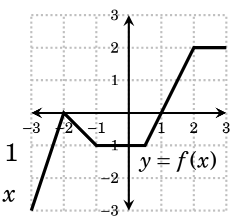

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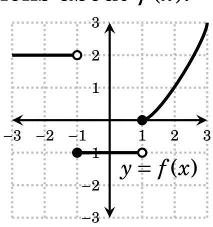
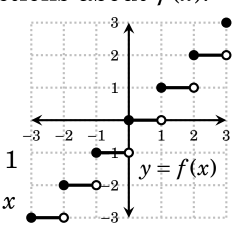


14. Answer the questions about $f(x)$.

a. $f(-3) =$
b. $f(0) =$
c. $f(2) =$
d. Solve: $f(x) = 1$
e. Solve: $f(x) = x$



15. Is the graph of $x^2 + xy + y^2 = 7$ also the graph of a function? Why or why not?

16. Is the graph of $x^4 = x^2 - y^2$ also the graph of a function? Why or why not?

17. Is the graph of $\dfrac{1}{x} + \dfrac{1}{y} = 2$ also the graph of a function? Why or why not?

18. Is the graph of $y^2 = x^3$ also the graph of a function? Why or why not?

19. Is the graph of $y^3 = x^2$ also the graph of a function? Why or why not?

20. Is the graph of $x^3 + y^3 - 4xy = 0$ also the graph of a function? Why or why not?

2.2 Domain and Range

Because a function's graph is the the set of all points $(x, f(x))$ for which x is in its domain, it follows that the idea of domain is an essential feature of a function. We now further explore and develop this idea, as well as a related feature of a function, called its *range*.

Definition 2.1 The **domain** of a function $f(x)$ is the set of all possible input values x. The function's **range** is the set of all possible output values $f(x)$, for which x is in the domain.

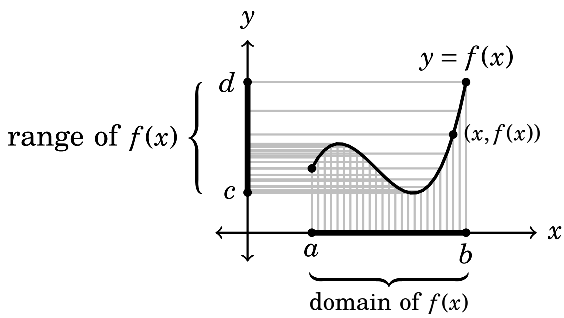
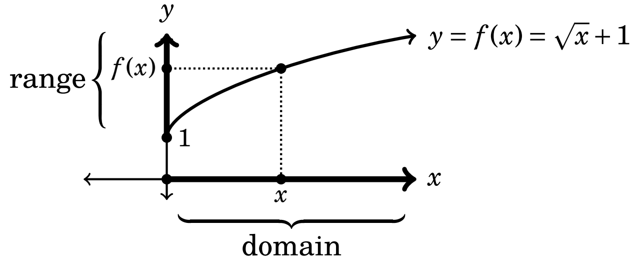


Figure 2.2. A function with domain $[a,b]$ and range $[c,d]$.

For example, Figure 2.2 shows a function whose domain is the segment on the number line between (and including) two numbers a and b. Recall that this segment is called a *closed interval* and is denoted $[a,b]$. Thus the domain of $f(x)$ is the interval $[a,b]$. Using the same notation, the range of $f(x)$ is the interval $[c,d]$.

This situation is typical. A function's domain and range tend to be intervals, or (as we shall see) combinations of several intervals.

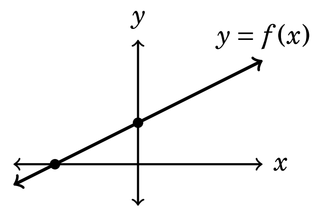
Example 2.1 Consider $f(x) = \sqrt{x}+1$. Because we can only (in this course) take square roots of numbers that are not negative, the domain of this function is the infinite interval $[0,\infty)$. Our experience with the graph (duplicated below) tells us that the range is the interval $[1,\infty)$. ✍



The next two examples illustrate an important point: A function's domain and range can be influenced by whether or not we are using the function to model some real-life situation.

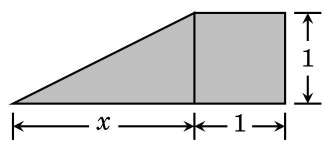
Example 2.2 Find the domain and range of the function $f(x) = \frac{1}{2}x + 1$.

The graph of this function is the straight line with slope 1/2 and y-intercept 1, graphed on the right. From this we infer that **the domain is all real numbers** and **the range is all real numbers**. That is, both the domain and range are $(-\infty, \infty)$. ✍



Example 2.3 This problem concerns the shape below, consisting of a one-by-one square with a triangle to one side. The triangle has height 1, but its base x can be any length we choose, so the area of this figure depends upon the length x.

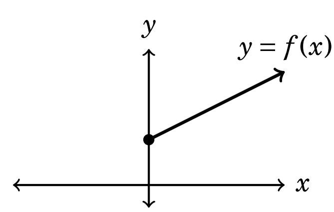
Let $f(x)$ be the function that gives the total area of this shaded region. Find an algebraic expression for $f(x)$, and find its domain and range.



To solve this, we first find an expression for $f(x)$. Notice that

$$\begin{aligned} f(x) &= \text{(area of triangle)} + \text{(area of square)} \\ &= \frac{1}{2}\text{(base)}\cdot\text{(height)} + 1\cdot 1 \\ &= \frac{1}{2}x\cdot 1 + 1. \end{aligned}$$

Thus $f(x) = \frac{1}{2}x + 1$, which is exactly the same function as in Example 2.2, above. The one difference is that x cannot be negative because it is the length of the triangle's base. The resulting graph of $f(x)$ is on the right.



From the graph we see that the domain of $f(x)$ is the interval $[0, \infty)$, which reflects the fact that the input x can be any (non-negative) length. The range is $[1, \infty)$, as $f(x)$ has to be at least 1 because the shaded region includes the square with area 1 square unit. ✍

Both of the Examples 2.2 and 2.3 involved a function of form $f(x) = \frac{1}{2}x + 1$. However in the first case the domain was all real numbers and in the second case the domain was all non-negative numbers. These examples highlight the fact that the *interpretation* of a function can influence what its domain and range are; in Example 2.3, we interpreted the input x as a length, so it couldn't be negative.

So in considering the domain of a function we need to think about how we are interpreting it. If it is just a generic function like $f(x) = \frac{1}{2}x + 1$, with no other meaning given, then we understand that the domain is all real numbers x for which $f(x)$ is defined.

In doing calculus we will often need to think about the concepts of domain and range. To hone our understanding we will now do a series of examples in which a function is given and we are asked to find the domain. However, in the future we will rarely have to find domains of functions explicitly, as is done here. The point of these examples is two-fold. First, they solidify the important concept of domain. Second, the solutions entail important techniques for solving equations and inequalities – techniques that will be used many, many times in this book, in other contexts.

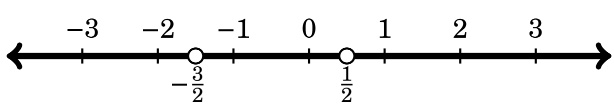
Example 2.4 Find the domain of the function $f(x) = \dfrac{1}{4x^2 + 4x - 3}$.

To solve this, let's think about what numbers are *not* in the domain. The only way that $f(x)$ might not be defined is if x is such that it causes division by zero, that is if x yields $4x^2 + 4x - 3 = 0$. To find such x, we just need to solve the equation $4x^2 + 4x - 3 = 0$. Factoring, we get

$$(2x - 1)(2x + 3) \quad = \quad 0.$$

Thus $x = 1/2$ and $x = -3/2$ are the values that make the denominator zero.

Therefore the domain consists of all real numbers *except* $1/2$ and $-3/2$. There are several ways to express this answer. One way is to write it as an unambiguous sentence, as we just did. Or we could describe it graphically, shading in all numbers on the number line that are in the domain.

Domain: 

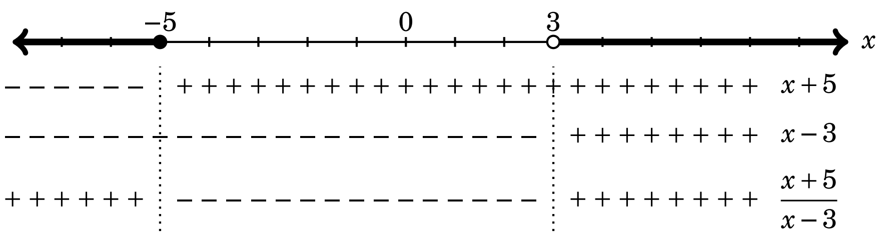
A third way to express the answer is to use interval notation. As seen above, the domain consists of three separate open intervals, $(-\infty, -3/2)$, $(-3/2, 1/2)$ and $(1/2, \infty)$. This can be written in the tidy mathematics expression $(-\infty, -3/2) \bigcup (-3/2, 1/2) \bigcup (1/2, \infty)$. (Recall that the symbol $\bigcup$ means *union*, so that if X and Y are sets of numbers, then $X \bigcup Y$ stands for the set of numbers belonging to either X or Y.)

Answer: The domain is $(-\infty, -3/2) \bigcup (-3/2, 1/2) \bigcup (1/2, \infty)$.

Example 2.5 Find the domain of the function $g(x) = \sqrt{\dfrac{x+5}{x-3}}$.

To begin, notice that $g(3) = \sqrt{\frac{3+5}{3-3}} = \sqrt{\frac{8}{0}}$ is not defined because it involves division by zero. Therefore 3 is not in the domain of $g(x)$.

There may be other x values that are not in the domain. For x to be in the domain, we require $\frac{x+5}{x-3} \geq 0$, so that in working out $g(x) = \sqrt{\frac{x+5}{x-3}}$ we don't take the square root of a negative number. If x is a value for which $\frac{x+5}{x-3} < 0$, then x is not in the domain of g. Therefore finding the domain involves finding all values of x for which $\frac{x+5}{x-3} > 0$. In other words, we must solve the inequality $\frac{x+5}{x-3} > 0$. You undoubtedly have solved such an inequality in prior courses. One technique involves examining the signs of the numerator $x+5$ and denominator $x-3$ at various points of the x-axis, and then reaching a conclusion about the sign of $\frac{x+5}{x-3}$. Carrying out this plan, we draw an x-axis (below). Note that $x+5$ is positive provided that x is greater than -5, and we tally this information by writing $x+5$ on the right and putting a line of + symbols under the x-axis and to the right of -5. We put − signs to the left of -5 because that is where $x+5$ is negative.



Next we consider the denominator $x-3$, which we write below the $x+5$. Note that $x-3$ is positive for $x > 3$ and negative for $x < 3$, so we draw a row of + and − symbols to indicate this. Now we can determine the sign of $\frac{x+5}{x-3}$. From the above diagram we see that for any x in the interval $(-\infty, -5)$, both $x+5$ and $x-3$ are negative, so the quotient $\frac{x+5}{x-3}$ is positive on $(-\infty, -5)$.

Also we see that for any x in the interval $(-5, 3)$ the numerator is positive and the denominator is negative. Therefore $\frac{x+5}{x-3}$ is negative on this interval. Finally, for any x in the interval $(-5, 3)$, both numerator and denominator are positive, so $\frac{x+5}{x-3} > 0$ here.

In summary, $\frac{x+5}{x-3}$ is positive on the intervals $(-\infty, -5)$ and $(3, \infty)$.

Answer: The domain of $g(x)$ is $(-\infty, -5] \cup (3, \infty)$. For clarity, this is shaded on the x-axis in the diagram. (Notice we include the endpoint -5 because $g(-5) = \frac{-5+5}{-5-3} = \sqrt{0} = 0$ is defined and therefore -5 is in the domain.) ✍

Example 2.6 Find the domain of the function $h(x) = \sqrt{\dfrac{1}{x} - 1}$.

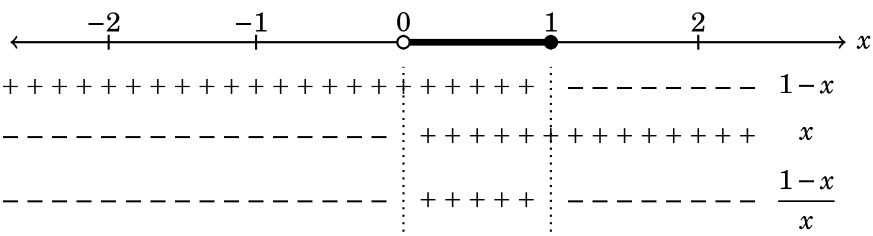
To begin, notice that $h(0)$ is not defined because it involves division by zero. Therefore 0 is not in the domain of the function.

Now let's find the other values of x that are not in the domain. The only other way that $h(x)$ might not be defined is if there is a negative value inside the radical: if x is such that $\frac{1}{x} - 1 < 0$. To find all such x we must solve the inequality $\frac{1}{x} - 1 < 0$, that is, find all values of x that make it true.

Let's use the approach in the previous example, where we examined the signs of the numerator and denominator of a quotient to analyze whether it was positive or negative. The only obstacle is that this time we are considering the sign of $\frac{1}{x} - 1$, which is not a quotient, but rather a quotient minus 1. We can overcome this by getting a common denominator and combining fractions.

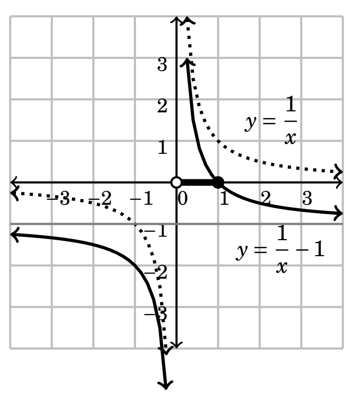
$$\frac{1}{x} - 1 = \frac{1}{x} - \frac{x}{x} = \frac{1-x}{x}$$

Thus, x is not in the domain whenever $\frac{1-x}{x} < 0$. To find such x for we chart the signs of $1-x$ and x separately.



The reveals that $\frac{1-x}{x}$ is positive whenever $0 < x < 1$. Thus the domain of $h(x)$ is the interval $(0,1]$. Note that 0 is not included in the domain because $h(0)$ is not defined, but 1 *is* included because $h(1) = 0$ *is defined.*

Here is another way to find the domain of $h(x) = \sqrt{\frac{1}{x} - 1}$. To determine the values of x that make $\frac{1}{x} - 1$ positive or negative, we could draw a graph of $y = \frac{1}{x} - 1$. To do this, we could start with the familiar graph $y = \frac{1}{x}$, sketched dotted. The graph of $y = \frac{1}{x} - 1$ is this dotted graph moved down one unit, shown bold. From this we see that $y = \frac{1}{x} - 1$ is positive only for those x in $(0,1]$. Thus the domain of $h(x)$ is the interval $(0,1]$.



This approach illustrates the usefulness of shifting graphs, a technique you've probably studied previously. Section 2.4 reviews this useful idea.

Our final example involves a function whose graph has a hole in it.

Example 2.7 Find the domain of $g(x) = \dfrac{x^2-1}{x-1}$. Draw the graph of $g(x)$.

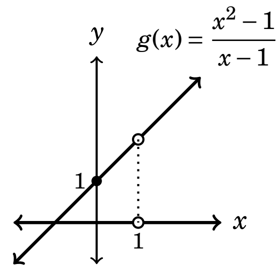
We can identify the domain immediately because $g(x)$ is defined for all values of x, except for $x = 1$, which makes the denominator zero. Therefore the domain is **all real numbers except 1.**

To draw the graph, it helps to notice that $g(x)$ simplifies as $g(x) = x+1$ by factoring the numerator and canceling, as follows:

$$g(x) \quad = \quad \frac{x^2-1}{x-1} \quad = \quad \frac{(x+1)(x-1)}{x-1} \quad = \quad (x+1)\frac{x-1}{x-1} \quad \underset{\uparrow}{=} \quad x+1.$$

$$\boxed{\textit{provided } x \neq 1}$$

In canceling the term $(x-1)$ we are in essence using the fact that the fraction $\dfrac{x-1}{x-1}$ equals 1, and therefore this fraction can be eliminated in the final step, above. The equation $\dfrac{x-1}{x-1} = 1$ is true for all values of x *except* for $x = 1$, because then it reduces to $\dfrac{0}{0}$, which is undefined. Thus cancellation of $(x-1)$ is valid only when $x \neq 1$.



In summary, $g(x) = x+1$ for all $x \neq 1$, and $g(x)$ is not defined when $x = 1$. The graph of $g(x)$ is therefore the straight line $y = x+1$, with a hole at the point $(1,2)$. Now, $y = x+1$ is easy to graph because it's a straight line with slope 1 and y-intercept 1. The final graph of $g(x)$ is shown above.

Functions whose graphs have holes arise very naturally in calculus. We will see many of them. ✍

Exercises for Section 2.2

For each of the following functions, find the domain and range.

1. $f(x) = x^2 - 1$ **2.** $f(x) = x^3 - 1$ **3.** $f(x) = \sqrt[3]{x}$ **4.** $g(w) = -\sqrt{-w}$

Find the domain of each of the following functions.

5. $f(x) = \dfrac{\sqrt{x}\sqrt{1-x}}{2x-1}$ **6.** $f(x) = \sqrt{\dfrac{x^2-7}{x-4}}$ **7.** $f(x) = \dfrac{\sqrt{-x}}{x^2-5}$ **8.** $f(x) = \dfrac{x}{\sqrt[4]{x-2}} + \dfrac{1}{x}$

2.3 An Inventory of Common Functions

Some functions (and families of functions) are so elemental and basic that they become part of our daily mathematical vocabulary. Here is a quick inventory. Other elemental functions, such as trigonometric, exponential and logarithmic functions, are treated in their own chapters.

We begin with linear functions. A **linear function** is a function that has the form $f(x) = mx + b$, where m and b are constants. The graph of this function is a straight line with slope m and y-intercept b.

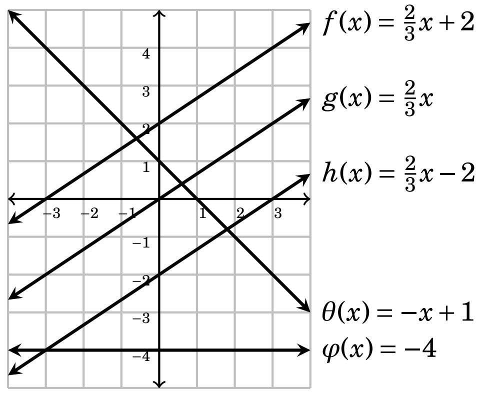


Figure 2.3. Some linear functions.

In $f(x) = mx + b$ it is of course possible that $m = 0$, giving the function $f(x) = b$. This is called a **constant function**; no matter what the input x is, the output is always the same number b. The graph of this function is a horizontal line (slope 0) passing through the point b on the y-axis. The constant function $\varphi(x) = -4$ is illustrated above. You could write is as $\varphi(x) = 0 \cdot x - 4$ and regard it as the rule *multiply x by zero and subtract* 4.

Before moving on we do one example.

Example 2.8 Suppose $f(x)$ is a linear function for which $f(1) = 4$ and $f(3) = 1$. Find $f(x)$.

To solve this, note that $f(1) = 4$ means that the point $(1,4)$ is on the graph of $f(x)$. Similarly, $f(3) = 1$ tells us that $(3,1)$ is on the graph. The graph is a straight line passing through these two points, so its slope is $\frac{\text{rise}}{\text{run}} = \frac{1-4}{3-1} = -\frac{3}{2}$. Using the point-slope formula with the point $(1,4)$, the equation for this line is

$$y - 4 = -\frac{3}{2}(x - 1),$$

simplifying to our final answer $f(x) = -\frac{3}{2}x + \frac{11}{2}$. ✍

A **power function** is one of form $f(x) = x^n$, where n is a positive integer. Figure 2.4 shows the first few examples, for n up to 5. It is important to *internalize* (not just memorize) these graphs. Take time to understand why the graphs look the way they do. Notice that when n is even x^n is positive for any x, so the graph lies above the x axis in those cases. By contrast, for odd n the value x^n is negative whenever x is negative; thus a portion of these graphs is below the x-axis.

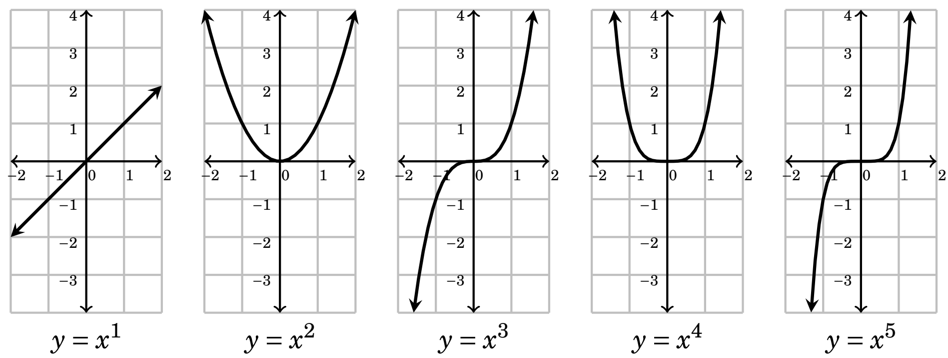


Figure 2.4. Power functions $f(x) = x^n$. In each case the domain is all real numbers, $\mathbb{R}$. If n is even the range is $[0,\infty)$. If n is odd the range is $\mathbb{R}$.

Reciprocals of power functions are functions of form $f(x) = \dfrac{1}{x^n}$. It is important to internalize these graphs too. Compare the graphs in Figure 2.5 with their companions in Figure 2.4.

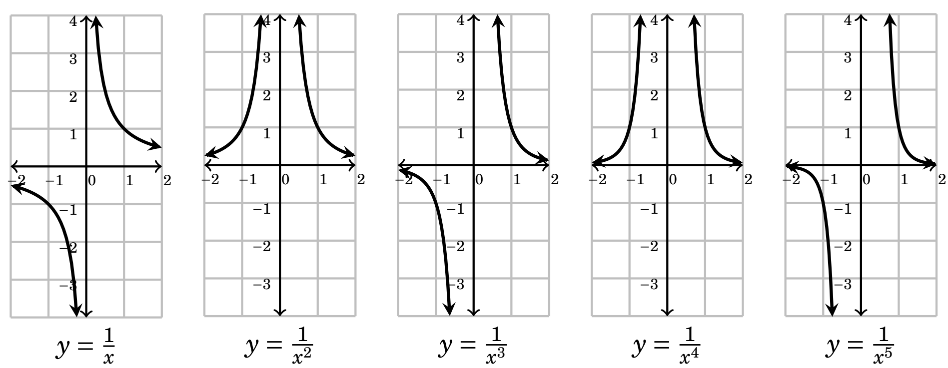


Figure 2.5. Reciprocals of power functions. In each case $f(x)$ becomes very close to 0 as x increases. And the closer x gets to 0, the bigger $f(x)$ becomes. In each case the domain is all real numbers except 0.

Continuing our inventory of functions, a **polynomial function** is a function given by a sum of multiples of power functions, plus a constant term (which could be 0). For example, $f(x) = x^4 - 2x^2 + \pi x + 2$ is a polynomial with constant term 2, and $g(x) = 5x^2 + 3x - 1$ is a polynomial with constant term -1. The **degree** of a polynomial is its highest power of x, so above $f(x)$ has degree 4 and $g(x)$ has degree 2. The linear function $h(x) = 3x + 7$ is a polynomial function of degree 1 (because $f(x) = 3x^1 + 7$) and constant term 7. The domain of a polynomial is all real numbers.

In a sense, the constant term of a polynomial can regarded as a multiple of a power of x. For example, in $h(x) = 3x + 7$ we have $7 = 7x^0$. Although this breaks down when $x = 0$ (as 0^0 is not defined) it is still a useful convention. For example, a constant function $f(x) = b = bx^0$ is a polynomial function of degree 0. Moreover, any linear function $h(x) = mx + b$ is a polynomial function with constant term b; its degree is 1 if $m \neq 0$, and 0 if $m = 0$.

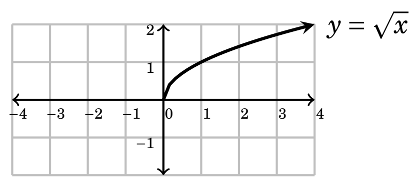
A **rational function** is a polynomial divided by a polynomial. (The word *rational* comes from *ratio*, so *rational* means *fractional*.) Here are some typical rational functions:

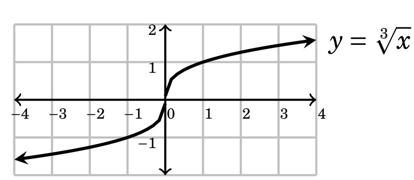
$$f(x) = \frac{x^2 + x + 1}{3x - 2}, \qquad g(x) = \frac{2x^3 + 4x^2 + 3x + \frac{1}{2}}{x^2 - 1} \qquad h(x) = \frac{1}{x^2 + 5}.$$

Note that any polynomial is also a rational function, because it is itself divided by the polynomial 1.

The power functions on the previous page are special cases of polynomial functions; their reciprocals are special cases of rational functions. Despite their simple forumlatiuon, in general the graphs of polynomials and rational functions can be quite complex.

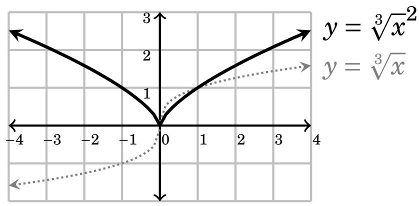
A **root function** is one having form $f(x) = \sqrt[n]{x}$. The root functions $f(x) = \sqrt{x}$ (that is, $f(x) = \sqrt[2]{x}$) and $f(x) = \sqrt[3]{x}$ are graphed below. Notice that if n is even, then $\sqrt[2]{x}$ is not defined for negative values of x, so in that case the domain is $[0, \infty)$. But this poses no problem if n is odd (e.g., $\sqrt[3]{-8} = -2$) so for odd n the domain of $f(x) = \sqrt[n]{x}$ is $\mathbb{R}$.





As $\sqrt[n]{x} = x^{\frac{1}{n}}$, we can regard root functions as variants of power functions.

Some functions have graphs with **cusps**, meaning sharp corners. For example, consider $f(x) = \sqrt[3]{x}^2$, the square of the function $y = \sqrt[3]{x}$ from the previous page. Its graph looks similar to that of $y = \sqrt[3]{x}$ (shown dotted, below) except that the y values are squared, hence any negative value $y = \sqrt[3]{x}$ becomes the positive number $y = \sqrt[3]{x}^2$. This function is graphed in bold below. Notice the cusp at the origin $(0,0)$.

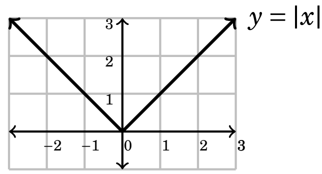


We will use this example (and variants of it) several times throughout this book. It turns out that cusps play an important role in certain applications.

Another function with a cusp is the **absolute value function** $f(x) = |x|$. Notice that if x is positive, then $f(x) = x$. If x is negative, then $f(x) = -x$, that is, $f(x)$ negates the negative x, turning it positive. For example, $f(-5) = -(-5) = 5$. Therefore the absolute value function can be thought of as a **piecewise function**, that is, a function in which the rule used depends upon the value of the input x.

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

The absolute value function is graphed below. Notice that for $x \geq 0$ the graph is $y = x$, which is a straight line with slope 1 and y-intercept 0. But when we look at the left side of the graph, where $x \leq 0$, the rule is $y = -x$, so we get the straight line with slope -1 and y-intercept 0.



Piecewise functions occur often enough to warrant at least one example. Consider the following function $y = g(t)$, whose domain is $[0, \infty)$.

$$g(t) = \begin{cases} 5t & \text{if } 0 \leq t \leq 4 \\ 20 - 16(t-4)^2 & \text{if } 4 < t < 5 \\ 4 & \text{if } 5 \leq t \end{cases}$$

This function is graphed below. Perhaps $g(t)$ represents the height at time t of an object that (beginning at time $t = 0$ seconds) is hoisted at a rate of five feet per second to the top of a 20 foot tower, then dropped off at time $t = 4$. One second later, at time $t = 5$, it lands on a four-foot high table and remains at rest there. The motion of this object is best described with a piecewise function.

We will use piecewise functions numerous times, especially in examples illustrating the idea of a *limit*, to be introduced later in this course.

Exercises for Section 2.3

1. Find the linear function $f(x)$ for which $f(-3) = 7$ and $f(2) = 1$.
2. Find the linear function $g(t)$ for which $g(4) = 3$ and $g(3) = 4$.
3. Find the linear function $h(x)$ with x-intercept 4 and y intercept 3.
4. Suppose $f(x)$ is a polynomial function of degree 2, for which $f(0) = 3$, $f(2) = 5$ and $f(-1) = 3$. Find $f(x)$.
5. Suppose $f(x)$ is a polynomial function of degree 2, for which $f(0) = 1$, $f(1) = 6$ and $f(2) = 17$. Find $f(x)$.
6. Suppose $f(x)$ is a polynomial function of degree 3, with x-intercepts -2, -1, 1, and y-intercept 2. Find $f(x)$.
7. Sketch the graph of the function $f(x) = \sqrt[5]{x^4}$.
8. Sketch the graph of the function $f(x) = \begin{cases} 3 & \text{if } x \le -1 \\ x^2+2 & \text{if } -1 < x < 1 \\ 3x & \text{if } 1 \le x. \end{cases}$
9. Sketch the graph of the function $g(w) = \begin{cases} -w & \text{if } w \le 0 \\ 1-w & \text{if } 0 < w < 1 \\ 2-w & \text{if } 1 \le w. \end{cases}$
10. Sketch the graph of the function $h(t) = \begin{cases} 2 & \text{if } t \le 0 \\ \sqrt{1-t^2}+1 & \text{if } 0 < t < 1 \\ 1 & \text{if } 1 \le t. \end{cases}$

2.4 Shifting Functions

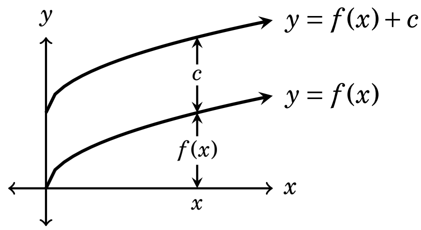
We now review some techniques that can help us sketch graphs of functions. Often a function can be understood as a modification of a simpler function like one of the basic functions (e.g., a root function) described in the previous section. In such a case the graph of the modified function is a transformation – or shifting – of the graph of the simpler familiar function.

First let's review three rules that involve vertical shifting of graphs. Suppose $f(x)$ is a function whose graph we know. The following boxes show how to obtain the graphs of $y = f(x)+c$, $y = cf(x)$ and $y = -f(x)$.

Suppose that c is a positive number.

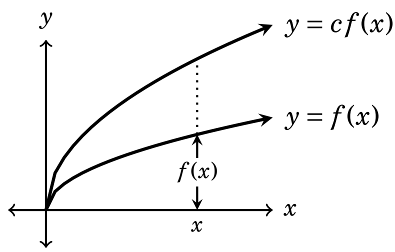
The graph of $y = f(x)+c$ is the graph of $y = f(x)$ moved up c units.

The graph of $y = f(x)-c$ is the graph of $y = f(x)$ moved down c units.

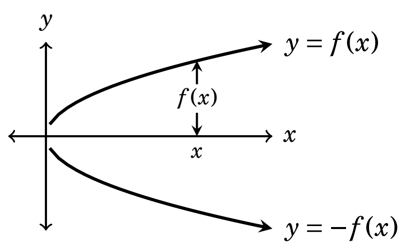


Suppose that c is a positive number.

The graph of $y = cf(x)$ is the graph of $y = f(x)$ scaled vertically by a factor of c.

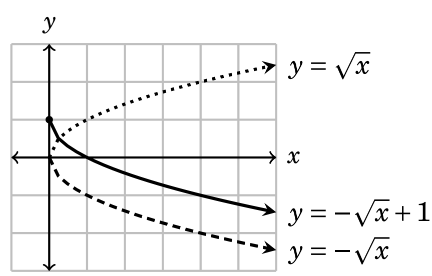


The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected across the x-axis.



Example 2.9 Sketch the graph of the function $f(x) = 1-\sqrt{x}$.

Start with the familiar graph $y = \sqrt{x}$, shown dotted. The graph of $y = -\sqrt{x}$ is this graph reflected across the x-axis, shown dashed. Finally, $y = -\sqrt{x}+1$ is the dashed graph moved up one unit, shown bold. This is the graph of $f(x) = 1-\sqrt{x}$. The others may be erased. ✍



The vertical shiftings of $f(x)$ reviewed on the previous page involved altering the output $y = f(x)$, changing it to $y = f(x) \pm c$, $y = cf(x)$ or $y = -f(x)$, for some positive number c. Next we look at corresponding *horizontal* shiftings of $f(x)$, which involve altering the *input* x, as $y = f(x+c)$, $y = f(cx)$ or $y = f(-x)$.

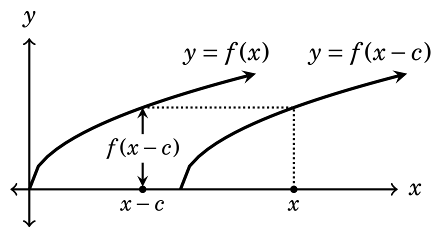
Suppose from $y = f(x)$ we make a new function $y = f(x-c)$. In words, the rule for $f(x-c)$ is *go back c units from x, to $x-c$, and compute $f(x-c)$*. Thus the graph of $y = f(x-c)$ is c units *ahead* (to the right) of that of $y = f(x)$.

This is summarized below, with two other types of horizontal shifting.

Suppose that c is a positive number.

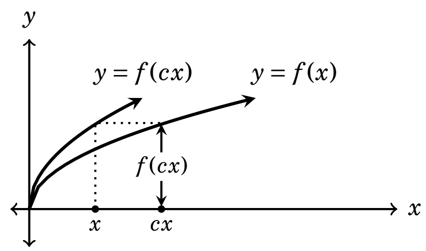
The graph of $y = f(x-c)$ is the graph of $y = f(x)$ moved c units right.

The graph of $y = f(x+c)$ is the graph of $y = f(x)$ moved c units left.

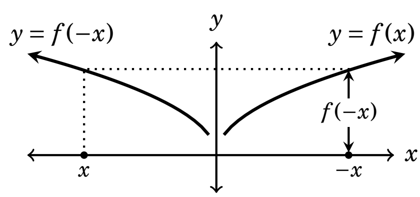


Suppose that c is a positive number.

The graph of $y = f(x)$ is the graph of $y = f(cx)$ scaled horizontally by a factor of c. In other words, the graph of $y = f(cx)$ is the graph of $y = f(x)$ scaled horizontally by a factor of $1/c$.

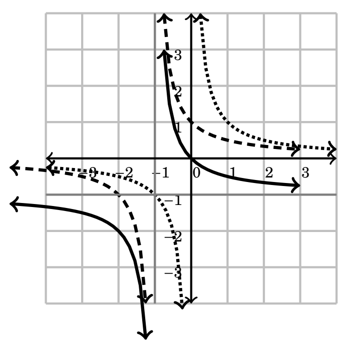


The graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected across the y-axis.



Example 2.10 Sketch the graph of the function $f(x) = \dfrac{1}{x+1} - 1$.

To solve this, start with the familiar graph $y = \dfrac{1}{x}$, shown dotted. The graph of $y = \dfrac{1}{x+1}$ is this graph moved one unit left, shown dashed. Finally, $f(x) = \dfrac{1}{x+1} - 1$ is the dashed graph moved down one unit, shown bold. ✍



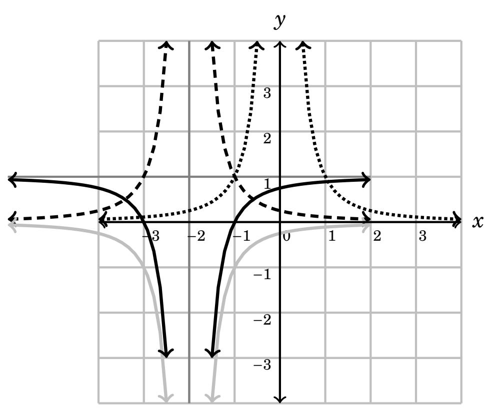
In using these shifting techniques you may sometimes need to algebraically massage a function into a form for which the shifting rules apply. The next example illustrates this. However, please be aware that graph shifting is a secondary technique that is occasionally used when we need to draw a quick sketch of a graph. If it involves too much work, then we reach the point of diminishing returns. If the next example seems too messy, then there is little harm in ignoring it.

Example 2.11 Sketch the graph of $f(x) = \dfrac{x^2+4x+3}{x^2+4x+4}$.

On the surface, this does not match any form to which a shifting rule might apply. But let's experiment with some algebraic manipulations to see if we can put it into such a form. Notice that the denominator factors as $x^2+4x+4 = (x+2)^2$, which, by itself, looks like a shifting of $y = x^2$ two units left. But that still leaves some x's in the numerator, so let's take care of that by writing the numerator as $x^2+4x+3 = x^2+4x+4-1$, in an attempt to mimic what's going on in the denominator. Now we have

$$\begin{aligned}
\frac{x^2+4x+3}{x^2+4x+4} &= \frac{(x^2+4x+4)-1}{x^2+4x+4} \\
&= \frac{(x+2)^2 \;-\; 1}{(x+2)^2} \\
&= \frac{(x+2)^2}{(x+2)^2} - \frac{1}{(x+2)^2} \\
&= 1 - \frac{1}{(x+2)^2}.
\end{aligned}$$

In other words, $f(x) = -\dfrac{1}{(x+2)^2} + 1$. This can be graphed with shifting operations. Start with the familiar graph of $y = \dfrac{1}{x^2}$ (see page 18), drawn dotted. Shift this two units left to get the graph of $y = \dfrac{1}{(x+2)^2}$ (dashed). Reflecting *this* across the x-axis gives the graph of $y = -\dfrac{1}{(x+2)^2}$ (gray). Moving this graph up one unit gives the graph of $f(x)$ (solid).



This completes the section's final example. Be sure to work some exercises for practice. ✍

Exercises for Section 2.4

Use the methods of this section to sketch graphs of the following functions.

1. $y = 2 - x^2$ **2.** $y = (x-3)^3 - 1$ **3.** $y = -\frac{1}{2}x^2$ **4.** $y = 3 - x^3$

5. $y = -\sqrt{x}$ **6.** $y = \sqrt{-x}$ **7.** $y = -\sqrt{-x}$ **8.** $y = \sqrt{x-4}$

9. $y = \sqrt[3]{-x}$ **10.** $y = \sqrt[3]{x+2} - 1$ **11.** $y = -\sqrt[3]{-x+2}$ **12.** $y = 4 - \sqrt[3]{x+2}$

13. $y = \dfrac{3}{x}$ **14.** $y = \dfrac{1}{x-3}$ **15.** $y = \dfrac{1}{x+3} + 2$ **16.** $y = \dfrac{1}{x^2 + 2x + 1}$

17. $y = |x-1| + 1$ **18.** $y = -|x-1|$ **19.** $y = -|x-1| + 1$ **20.** $y = 2|x-1| + 1$

2.5 Combining Functions

Given two functions, we can combine them in various ways to create new functions. For example, from $f(x)$ and $g(x)$, we can form the new function

$$h(x) = f(x) + g(x).$$

Sometimes this function, the sum of $f(x)$ and $g(x)$, is denoted as $(f+g)(x)$:

$$(f+g)(x) = f(x) + g(x).$$

Here the grouping of symbols $(f+g)$ is the *name* of a new function formed from f and g. We can combine f and g with any arithmetic operation.

$(f+g)(x) = f(x) + g(x)$	$(fg)(x) = f(x) \cdot g(x)$
$(f-g)(x) = f(x) - g(x)$	$\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$

Example 2.12 Suppose $f(x) = x^2 + 3x + 1$ and $g(x) = x - 4$. Then

$$\begin{array}{rclcl}
(f+g)(x) & = & (x^2+3x+1) + (x-4) & = & x^2 + 4x - 3, \\
(f-g)(x) & = & (x^2+3x+1) - (x-4) & = & x^2 + 2x + 5, \\
(fg)(x) & = & (x^2+3x+1)(x-4) & = & x^3 - x^2 - 6x - 4, \\
\left(\dfrac{f}{g}\right)(x) & = & \dfrac{x^2+3x+1}{x-4}. & &
\end{array}$$

In this way complex functions can be built up from simpler functions. Just as importantly, these operations can break down complex functions into simpler functions. The function $y = \frac{x^2+3x+1}{x-4}$ above is broken down into a quotient of simpler functions $f(x) = x^2 + 3x + 1$ and $g(x) = x - 4$. ✍

Another important way of combining functions is through *composition*. Composition combines $f(x)$ and $g(x)$ by plugging $g(x)$ into f to get $f(g(x))$.

The **composition** of two functions $f(x)$ and $g(x)$ is a new function, called $f \circ g$, and defined as $(f \circ g)(x) = f\big(g(x)\big)$.

Example 2.13 Suppose that $f(x) = x + \dfrac{1}{x}$ and $g(x) = \sqrt{x}$. Then

$$\begin{aligned}
(f \circ g)(x) &= f\big(g(x)\big) &&= g(x) + \frac{1}{g(x)} &&= \sqrt{x} + \frac{1}{\sqrt{x}} \\
(g \circ f)(x) &= g\big(f(x)\big) &&= \sqrt{f(x)} &&= \sqrt{x + \frac{1}{x}} \\
(f \circ f)(x) &= f\big(f(x)\big) &&= f(x) + \frac{1}{f(x)} &&= \frac{1}{x} + \frac{1}{x + \frac{1}{x}} &&= \frac{1}{x} + \frac{x}{x^2 + 1}.
\end{aligned}$$

This example reveals that in general $f\big(g(x)\big) \neq g\big(f(x)\big)$. ✍

Take note that composition $(f \circ g)(x) = f\big(g(x)\big)$ is *very* different from multiplication $f(x) \cdot g(x)$. Do not confuse the two. The product $f(x) \cdot g(x)$ of functions is a new function whose rule is this: *given an input x, compute $f(x)$ and $g(x)$; then multiply these numbers*. Composition $f(g(x))$ produces a new function whose rule is: *given an input x, compute $g(x)$ and then plug this number into $f(x)$*.

Compositions can also be formed with three or more functions.

Example 2.14 Suppose $f(x) = x^2 + x$, $g(x) = \sqrt{x}$ and $h(x) = 5x + 7$. Then:

$$\begin{aligned}
(f \circ g \circ h)(x) &= f\big(g(h(x))\big) \\
&= \big(g\big(h(x)\big)\big)^2 + g\big(h(x)\big) \\
&= \sqrt{h(x)}^2 + \sqrt{h(x)} \\
&= \sqrt{5x+7}^2 + \sqrt{5x+7} \\
&= 5x + 7 + \sqrt{5x+7}.
\end{aligned}$$

As described above, composition combines two functions to produce a third function, which is typically more complex than the two functions we started with. The reverse of this process is vitally important: Write a complicated function as the composition of two simpler functions.

Example 2.15 Write $h(x) = \sqrt{x^2 + 3x - 4}$ as a composition of two simpler functions.

Put $f(x) = \sqrt{x}$ and $g(x) = x^2 + 3x - 4$. Then we have $h(x) = f\big(g(x)\big)$. ✍

Exercises for Section 2.5

A. Find $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$, $\left(\frac{f}{g}\right)(x)$, $(f\circ g)(x)$ and $(g\circ h)(x)$.

1. $f(x)=x^2-1$ and $g(x)=\sqrt{x}$

2. $f(x)=\dfrac{1}{x}$ and $g(x)=\dfrac{1}{3x}$

3. $f(x)=x^2$ and $g(x)=5x$

4. $f(x)=5+x$ and $g(x)=2-x$

5. $f(x)=1+3x$ and $g(x)=5+x^2$

6. $f(x)=x+\cos(x)$ and $g(x)=x-\cos(x)$

B. Find $(f\circ g)(x)$ and $(g\circ h)(x)$.

7. $f(x)=\dfrac{x+2}{1-x}$ and $g(x)=x+\sqrt{x}-1$

8. $f(x)=\dfrac{1}{1-x}$ and $g(x)=\dfrac{1}{x}$

9. $f(x)=x^2+1$ and $g(x)=3-x$

10. $f(x)=x\cos(x)$ and $g(x)=x+x^2$

11. $f(x)=x^2-1$ and $g(x)=\sqrt{x+1}$

12. $f(x)=x^3+5$ and $g(x)=\sqrt[3]{x-5}$

C. Write the function as a composition of two simpler functions.

13. $\cos(x^2+2)-\sin(x^2+1)$

14. $\sqrt{\dfrac{x^2-7}{x-4}}$

15. $h(x)=\dfrac{\sqrt{x^2+1}}{x^2}$

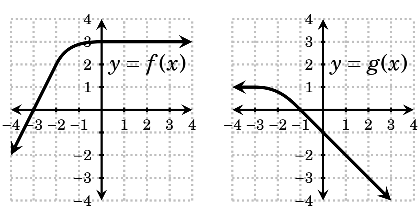
16. $\dfrac{1}{x^5-4x^2+3x+1}$

17. $2^{3\sqrt{x}}$

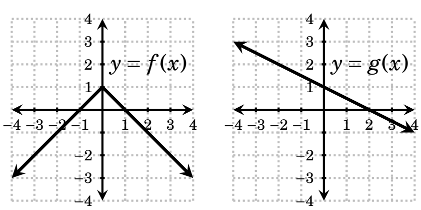
18. $\sqrt{x}+\dfrac{1}{\sqrt{x}}$

D. Supply the indicated information about the functions graphed below.

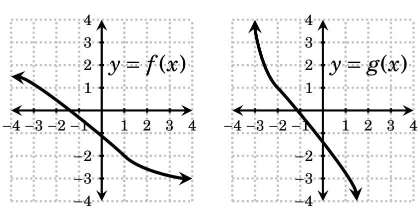
19. Find $g(f(1))$, $g(f(-2))$, $g(f(-3))$, $f(g(2))$ and $f(g(-1))$.



20. Find $g(f(1))$, $g(f(-2))$, $g(f(-3))$, $f(g(2))$ and $f(g(-1))$.



21. Find $g(f(1))$, $g(f(-2))$, $g(f(-3))$, $f(g(-2))$ and $f(g(-3))$.



22. Find $g(f(0))$, $g(f(2))$, $g(f(4))$, $f(g(-3))$ and $f(g(4))$.