Chapter 19

Cardinality of Sets

This chapter is all about cardinality of sets. At first this looks like a very simple concept. To find the cardinality of a set, just count its elements. If $A = \{a, b, c, d\}$, then $|A| = 4$; if $B = \{n \in \mathbb{Z} : -5 \leq n \leq 5\}$, then $|B| = 11$. In this case $|A| < |B|$. What could be simpler than that?

Actually, the idea of cardinality becomes quite subtle when the sets are infinite. The main point of this chapter is to explain how there are numerous different kinds of infinity, and some infinities are bigger than others. Two sets A and B can both have infinite cardinality, yet $|A| < |B|$.

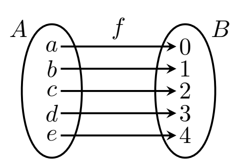
19.1 Sets with Equal Cardinalities

We begin with a discussion of what it means for two sets to have the same cardinality. Up until this point we've said $|A| = |B|$ if A and B have the same number of elements: Count the elements of A, then count the elements of B. If you get the same number, then $|A| = |B|$.

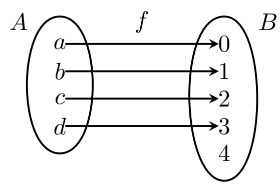
Although this is a fine strategy if the sets are finite (and not too big!), it doesn't apply to infinite sets because we'd never be done counting their elements. We need a new approach that applies to both finite and infinite sets. Here it is:

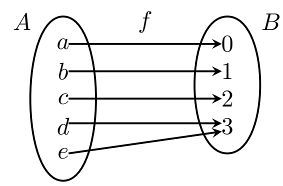
Definition 19.1. Sets A and B have the **same cardinality**, written $|A| = |B|$, if there is a bijective function $f : A \to B$. If no such function exists, then the sets have **unequal cardinalities**, that is, $|A| \neq |B|$.

The picture below illustrates this. There is a bijective function $f : A \to B$, so $|A| = |B|$. The function f matches up A with B. Think of f as describing how to overlay A onto B so that they fit together perfectly.



On the other hand, if A and B are as indicated in either of the following figures, then there can be no bijection $f : A \to B$. (The best we can do is a function that is either injective or surjective, but not both). Therefore the definition says $|A| \neq |B|$ in these cases.





Example 19.1. Sets $A = \{n \in \mathbb{Z} : 0 \leq n \leq 5\}$ and $B = \{n \in \mathbb{Z} : -5 \leq n \leq 0\}$ have the same cardinality because $f(n) = -n$ is a bijective function $f : A \to B$. ✍

Several comments are in order. First, if $|A| = |B|$, there can be *lots* of bijective functions from A to B. We only need to find one of them in order to conclude $|A| = |B|$. Second, as bijective functions play such a big role here, we use the word **bijection** to mean *bijective function.* Thus the function $f(n) = -n$ from Example 19.1 is a bijection. Also, an injective function is called an **injection** and a surjective function is called a **surjection**.

We emphasize and reiterate that Definition 19.1 applies to finite as well as infinite sets. If A and B are infinite, then $|A| = |B|$ provided there exists a bijection $f : A \to B$. If no such bijection exists, then $|A| \neq |B|$.

Example 19.2. This example shows that $|\mathbb{N}| = |\mathbb{Z}|$. To see why this is true, notice that the following table describes a bijection $f : \mathbb{N} \to \mathbb{Z}$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15 ...
$f(n)$	0	1	-1	2	-2	3	-3	4	-4	5	-5	6	-6	7	-7 ...

Notice that f is described in such a way that it is both injective and surjective. Every integer appears exactly once on the infinitely long second row. Thus, according to the table, given any $b \in \mathbb{Z}$ there is some natural number n with $f(n) = b$, so f is surjective. It is injective because the way the table is constructed forces $f(m) \neq f(n)$ whenever $m \neq n$. Because of this bijection $f : \mathbb{N} \to \mathbb{Z}$, we must conclude from Definition 19.1 that $|\mathbb{N}| = |\mathbb{Z}|$. ✍

Example 19.2 may seem slightly unsettling. On one hand it makes sense that $|\mathbb{N}| = |\mathbb{Z}|$ because $\mathbb{N}$ and $\mathbb{Z}$ are both infinite, so their cardinalities are both “infinity.” On the other hand, $\mathbb{Z}$ may seem twice as large as $\mathbb{N}$ because $\mathbb{Z}$ has all the negative integers as well as the positive ones. Definition 19.1 settles the issue. Because the bijection $f : \mathbb{N} \to \mathbb{Z}$ matches up $\mathbb{N}$ with $\mathbb{Z}$, it follows that $|\mathbb{N}| = |\mathbb{Z}|$. We summarize this with a theorem.

Theorem 19.1. *There exists a bijection $f : \mathbb{N} \to \mathbb{Z}$. Therefore $|\mathbb{N}| = |\mathbb{Z}|$.*

The fact that $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality might prompt us compare the cardinalities of other infinite sets. How, for example, do $\mathbb{N}$ and $\mathbb{R}$ compare? Let's turn our attention to this.

In fact, $|\mathbb{N}| \neq |\mathbb{R}|$. This was first recognized by Georg Cantor (1845–1918), who devised an ingenious argument to show that there are no surjective functions $f : \mathbb{N} \to \mathbb{R}$. (This in turn implies that there can be no bijections $f : \mathbb{N} \to \mathbb{R}$, so $|\mathbb{N}| \neq |\mathbb{R}|$ by Definition 19.1.)

We now describe Cantor's argument for why there are no surjections $f : \mathbb{N} \to \mathbb{R}$. We will reason informally, rather than writing out an exact proof. Take any arbitrary function $f : \mathbb{N} \to \mathbb{R}$. Here's why f can't be surjective:

Imagine making a table for f, where values of n in $\mathbb{N}$ are in the left-hand column and the corresponding values $f(n)$ are on the right. The first few entries might look something as follows. In this table, the real numbers $f(n)$ are written with all their decimal places trailing off to the right. Thus, even though $f(1)$ happens to be the real number 0.4, we write it as $0.40000000\ldots$., etc.

n	$f(n)$
1	0 . 4 0 0 0 0 0 0 0 0 0 0 0 0 0...
2	8 . 5 0 0 6 0 7 0 8 6 6 6 9 0 0...
3	7 . 5 0 5 0 0 9 4 0 0 4 4 1 0 1...
4	5 . 5 0 7 0 4 0 0 8 0 4 8 0 5 0...
5	6 . 9 0 0 2 6 0 0 0 0 0 0 5 0 6...
6	6 . 8 2 8 0 9 5 8 2 0 5 0 0 2 0...
7	6 . 5 0 5 0 5 5 5 0 6 5 5 8 0 8...
8	8 . 7 2 0 8 0 6 4 0 0 0 0 4 4 8...
9	0 . 5 5 0 0 0 0 8 8 8 8 0 0 7 7...
10	0 . 5 0 0 2 0 7 2 2 0 7 8 0 5 1...
11	2 . 9 0 0 0 0 8 8 0 0 0 0 9 0 0...
12	6 . 5 0 2 8 0 0 0 8 0 0 9 6 7 1...
13	8 . 8 9 0 0 8 0 2 4 0 0 8 0 5 0...
14	8 . 5 0 0 0 8 7 4 2 0 8 0 2 2 6...
$\vdots$	$\vdots$

There is a diagonal shaded band in the table. For each $n \in \mathbb{N}$, this band covers the n^{th} decimal place of $f(n)$:

The 1st decimal place of $f(1)$ is the 1st entry on the diagonal.
The 2nd decimal place of $f(2)$ is the 2nd entry on the diagonal.
The 3rd decimal place of $f(3)$ is the 3rd entry on the diagonal.
The 4th decimal place of $f(4)$ is the 4th entry on the diagonal, etc.

The diagonal helps us construct a number $b \in \mathbb{R}$ that is unequal to any $f(n)$. Just let the nth decimal place of b differ from the nth entry of the diagonal. Then the nth decimal place of b differs from the nth decimal place of $f(n)$. To be definite, define b to be the positive number less than 1 whose nth decimal place is 0 if the nth decimal place of $f(n)$ is not 0, and whose nth decimal place is 1 if the nth decimal place of $f(n)$ *is* 0. Thus, for the function f from the above table, we have

$$b = 0.01010001001000\ldots$$

and b has been defined so that, for any $n \in \mathbb{N}$, its nth decimal place is unequal to the nth decimal place of $f(n)$. Therefore $f(n) \neq b$ for every natural number n, meaning f is not surjective.

Since this argument applies to *any* function $f : \mathbb{N} \to \mathbb{R}$ (not just the one in the above example) we conclude that there exist no bijections $f : \mathbb{N} \to \mathbb{R}$, so $|\mathbb{N}| \neq |\mathbb{R}|$ by Definition 19.1. We summarize this as a theorem.

Theorem 19.2. *There exists no bijection* $f : \mathbb{N} \to \mathbb{R}$. *Therefore* $|\mathbb{N}| \neq |\mathbb{R}|$.

This is our first indication of how there are different kinds of infinities. Both $\mathbb{N}$ and $\mathbb{R}$ are infinite sets, yet $|\mathbb{N}| \neq |\mathbb{R}|$. We will continue to develop this theme throughout this chapter. The next example shows that the intervals $(0, \infty)$ and $(0, 1)$ on $\mathbb{R}$ have the same cardinality.

Example 19.3. Show that $|(0, \infty)| = |(0, 1)|$. To accomplish this, we need to show that there is a bijection $f : (0, \infty) \to (0, 1)$. We describe this function geometrically. Consider the interval $(0, \infty)$ as the positive x-axis of $\mathbb{R}^2$. Let the interval $(0, 1)$ be on the y-axis as illustrated in Figure 19.1, so that $(0, 1)$ is perpendicular to $(0, \infty)$.

The figure also shows a point $P = (-1, 1)$. Define $f(x)$ to be the point on $(0, 1)$ where the line from P to $x \in (0, \infty)$ intersects the y-axis. By similar triangles,

$$\frac{1}{x+1} = \frac{f(x)}{x}, \quad \text{and therefore} \quad f(x) = \frac{x}{x+1}.$$

If it is not clear from the figure that $f : (0, \infty) \to (0, 1)$ is bijective, you can verify it using the techniques from Section 18.2. (Exercise 16, below.) ✍

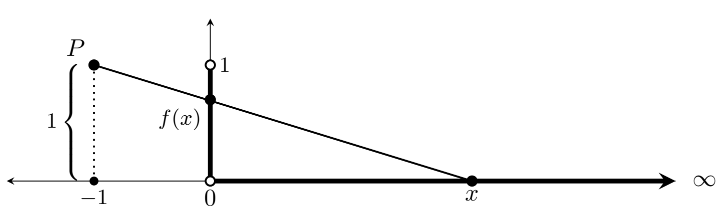


Fig. 19.1 A bijection $f : (0, \infty) \to (0, 1)$

It is important to note that equality of cardinalities is an equivalence relation on sets: it is reflexive, symmetric and transitive. Let us confirm this. Given a set A, the identity function $A \to A$ is a bijection, so $|A| = |A|$. (This is the reflexive property.) For the symmetric property, if $|A| = |B|$, then there is a bijection $f : A \to B$, and its inverse is a bijection $f^{-1} : B \to A$, so $|B| = |A|$. For transitivity, suppose $|A| = |B|$ and $|B| = |C|$. Then there are bijections $f : A \to B$ and $g : B \to C$. The composition $g \circ f : A \to C$ is a bijection (Theorem 18.3), so $|A| = |C|$.

The transitive property can be useful. If, in trying to show two sets A and C have the same cardinality, we can produce a third set B for which $|A| = |B|$ and $|B| = |C|$, then transitivity assures us that indeed $|A| = |C|$. The next example uses this idea.

Example 19.4. Show that $|\mathbb{R}| = |(0, 1)|$.

Because of the bijection $g : \mathbb{R} \to (0, \infty)$ where $g(x) = 2^x$, we have $|\mathbb{R}| = |(0, \infty)|$. Also, Example 19.3 shows that $|(0, \infty)| = |(0, 1)|$. Therefore $|\mathbb{R}| = |(0, 1)|$. ✍

So far in this chapter we have declared that two sets have "the same cardinality" if there is a bijection between them. They have "different cardinalities" if there exists no bijection between them. Using this idea, we showed that $|\mathbb{Z}| = |\mathbb{N}| \neq |\mathbb{R}| = |(0, \infty)| = |(0, 1)|$. So, we have a means of determining when two sets have the same or different cardinalities. But we have neatly avoided saying exactly what cardinality *is*. For example, we can say that $|\mathbb{Z}| = |\mathbb{N}|$, but what exactly *is* $|\mathbb{Z}|$, or $|\mathbb{N}|$? What exactly *are* these things that are equal? Certainly not numbers, for they are too big. And saying they are "infinity" is not accurate, because we now know that there are different types of infinity. So just what kind of mathematical entity is $|\mathbb{Z}|$? In general, given a set X, exactly what *is* its cardinality $|X|$?

This is a lot like asking what a number is. A number, say 5, is an abstraction, not a physical thing. Early in life we instinctively grouped together certain sets of things (five apples, five oranges, etc.) and conceived of 5 as the thing common to all such sets. In a very real sense, the number 5 is an abstraction of the fact that any two of these sets can be matched up via a bijection. That is, it can be identified with a certain equivalence class of sets under the "*has the same cardinality as*" relation. (Recall that this is an equivalence relation.) This is easy to grasp because our sense of numeric quantity is so innate. But in exactly the same way we can say that the cardinality of a set X is what is common to all sets that can be matched to X via a bijection. This may be harder to grasp, but it is really no different from the idea of the magnitude of a (finite) number.

In fact, we could be concrete and define $|X|$ to be the equivalence class of all sets whose cardinality is the same as that of X. This has the advantage of giving an explicit meaning to $|X|$. But there is no harm in taking the intuitive approach and just interpreting the cardinality $|X|$ of a set X to be a measure the "size" of X. The point of this section is that we have a means of deciding whether two sets have the same size or different sizes.

Exercises for Section 19.1

A. Show that the two given sets have equal cardinality by describing a bijection from one to the other. Describe your bijection with a formula (not as a table).

1. $\mathbb{R}$ and $(0,\infty)$

2. $\mathbb{R}$ and $(\sqrt{2},\infty)$

3. $\mathbb{R}$ and $(0,1)$

4. The set of even integers and the set of odd integers

5. $A=\{3k:k\in\mathbb{Z}\}$ and $B=\{7k:k\in\mathbb{Z}\}$

6. $\mathbb{N}$ and $S=\left\{\frac{\sqrt{2}}{n} : n\in\mathbb{N}\right\}$

7. $\mathbb{Z}$ and $S=\left\{\ldots,\frac{1}{8},\frac{1}{4},\frac{1}{2},1,2,4,8,16,\ldots\right\}$

8. $\mathbb{Z}$ and $S=\{x\in\mathbb{R}:\sin x=1\}$

9. $\{0,1\}\times\mathbb{N}$ and $\mathbb{N}$

10. $\{0,1\}\times\mathbb{N}$ and $\mathbb{Z}$

11. $[0,1]$ and $(0,1)$

12. $\mathbb{N}$ and $\mathbb{Z}$ (See Exercise 18 of Section 18.2.)

13. $\mathscr{P}(\mathbb{N})$ and $\mathscr{P}(\mathbb{Z})$ (Suggestion: use Exercise 12, above.)

14. $\mathbb{N}\times\mathbb{N}$ and $\{(n,m)\in\mathbb{N}\times\mathbb{N}:n\le m\}$

B. Answer the following questions concerning bijections from this section.

15. Find a formula for the bijection f in Example 19.2 (page 428).

16. Verify that the function f in Example 19.3 is a bijection.

19.2 Countable and Uncountable Sets

Let's summarize the main points from the previous section.

- $|A|=|B|$ if and only if there exists a bijection $A\to B$.
- $|\mathbb{N}|=|\mathbb{Z}|$ because there exists a bijection $\mathbb{N}\to\mathbb{Z}$.
- $|\mathbb{N}|\neq|\mathbb{R}|$ because there exists *no* bijection $\mathbb{N}\to\mathbb{R}$.

Thus, even though $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ are all infinite sets, their cardinalities are not all the same. The sets $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality, but $\mathbb{R}$'s cardinality is different from that of both the other sets. This means infinite sets can have different sizes. We now make some definitions to put words and symbols to this phenomenon.

In a certain sense you can count the elements of $\mathbb{N}$; you can count its elements off as $1,2,3,4,\ldots$, but you'd have to continue this process forever to count the whole set. Thus we will call $\mathbb{N}$ a *countably infinite set*, and the same term is used for any set whose cardinality equals that of $\mathbb{N}$.

Definition 19.2. Suppose A is a set. Then A is **countably infinite** if $|\mathbb{N}|=|A|$, that is, if there exists a bijection $\mathbb{N}\to A$. The set A is **uncountable** if A is infinite and $|\mathbb{N}|\neq|A|$, that is, if A is infinite and there exists *no* bijection $\mathbb{N}\to A$.

Thus $\mathbb{Z}$ is countably infinite but $\mathbb{R}$ is uncountable. This section deals mainly with countably infinite sets. Uncountable sets are treated later.

If A is countably infinite, then $|\mathbb{N}|=|A|$, so there is a bijection $f:\mathbb{N}\to A$. You can think of f as "counting" the elements of A. The first element of A is $f(1)$, followed by $f(2)$, then $f(3)$ and so on. It makes sense to think of a countably infinite set as the smallest type of infinite set, because if the counting process stopped, the set would be finite, not infinite; a countably infinite set has the fewest elements that a set can have and still be infinite. It is common to reserve the special symbol $\aleph_0$ to stand for the cardinality of countably infinite sets.

Definition 19.3. The cardinality of the natural numbers $\mathbb{N}$ is denoted by $\aleph_0$. That is, $|\mathbb{N}| = \aleph_0$. Thus any countably infinite set has cardinality $\aleph_0$.

(The symbol $\aleph$, pronounced "aleph," is the first letter in the Hebrew alphabet. The symbol $\aleph_0$ is pronounced "aleph naught.") The summary of facts at the beginning of this section shows $|\mathbb{Z}| = \aleph_0$ and $|\mathbb{R}| \neq \aleph_0$.

Example 19.5. Let $E = \{2k : k \in \mathbb{Z}\}$ be the set of even integers. The function $f : \mathbb{Z} \to E$ defined as $f(n) = 2n$ is easily seen to be a bijection, so $|\mathbb{Z}| = |E|$. Thus, as $|\mathbb{N}| = |\mathbb{Z}| = |E|$, the set E is countably infinite and $|E| = \aleph_0$. ✍

Here is a significant fact: The elements of any countably infinite set A can be written in an infinitely long list $a_1, a_2, a_3, a_4, \ldots$ that begins with some element $a_1 \in A$ and includes every element of A. For example, the set E in the above example can be written in list form as $0, 2, -2, 4, -4, 6, -6, 8, -8, \ldots$ The reason that this can be done is as follows. Since A is countably infinite, Definition 19.2 says there is a bijection $f : \mathbb{N} \to A$. This allows us to list out the set A as an infinite list $f(1), f(2), f(3), f(4), \ldots$ Conversely, if the elements of A can be written in list form as $a_1, a_2, a_3, \ldots$, then the function $f : \mathbb{N} \to A$ defined as $f(n) = a_n$ is a bijection, so A is countably infinite. We summarize this as follows.

Theorem 19.3. *A set A is countably infinite if and only if its elements can be arranged in an infinite list $a_1, a_2, a_3, a_4, \ldots$*

For example, say P is the set of all prime numbers. Since we can list its elements as $2, 3, 5, 7, 11, 13, \ldots$, it follows that the set P is countably infinite.

By Theorem 19.3, we can interpret the fact that the set $\mathbb{R}$ is not countably infinite as meaning that it is impossible to write out all the elements of $\mathbb{R}$ in an infinite list. (After all, we tried to do that in the table on page 429, and failed!)

This raises a question. Can we write out all the elements of $\mathbb{Q}$ in an infinite list? In other words, is the set $\mathbb{Q}$ of rational numbers countably infinite or uncountable? If you start plotting the rational numbers on the number line, they seem to mostly fill up $\mathbb{R}$. Sure, some numbers such as $\sqrt{2}$, π and e will not be plotted, but the dots representing rational numbers seem to predominate. We might thus expect $\mathbb{Q}$ to be uncountable. However, it is a surprising fact that $\mathbb{Q}$ is countable. The proof presented below arranges all the rational numbers in an infinitely long list.

Theorem 19.4. *The set $\mathbb{Q}$ of rational numbers is countably infinite.*

Proof. To prove this, we just need to show how to write the set $\mathbb{Q}$ in list form. Begin by arranging all rational numbers the infinite array shown below. The top row has a list of all integers, beginning with 0, then alternating signs as they increase. Each column headed by an integer k contains all the fractions (in reduced form) with numerator k. For example, the column headed by 2 contains the fractions $\frac{2}{1}, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \ldots$, etc. It does not contain $\frac{2}{2}$, $\frac{2}{4}$, $\frac{2}{6}$, etc., because those are not reduced, and in fact their reduced forms appear in the column headed by 1. You should examine this table and convince yourself that it contains all rational numbers in $\mathbb{Q}$.

0	1	-1	2	-2	3	-3	4	-4	5	-5	$\cdots$
$\frac{0}{1}$	$\frac{1}{1}$	$\frac{-1}{1}$	$\frac{2}{1}$	$\frac{-2}{1}$	$\frac{3}{1}$	$\frac{-3}{1}$	$\frac{4}{1}$	$\frac{-4}{1}$	$\frac{5}{1}$	$\frac{-5}{1}$	$\cdots$
	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{2}{3}$	$\frac{-2}{3}$	$\frac{3}{2}$	$\frac{-3}{2}$	$\frac{4}{3}$	$\frac{-4}{3}$	$\frac{5}{2}$	$\frac{-5}{2}$	$\cdots$
	$\frac{1}{3}$	$\frac{-1}{3}$	$\frac{2}{5}$	$\frac{-2}{5}$	$\frac{3}{4}$	$\frac{-3}{4}$	$\frac{4}{5}$	$\frac{-4}{5}$	$\frac{5}{3}$	$\frac{-5}{3}$	$\cdots$
	$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{2}{7}$	$\frac{-2}{7}$	$\frac{3}{5}$	$\frac{-3}{5}$	$\frac{4}{7}$	$\frac{-4}{7}$	$\frac{5}{4}$	$\frac{-5}{4}$	$\cdots$
	$\frac{1}{5}$	$\frac{-1}{5}$	$\frac{2}{9}$	$\frac{-2}{9}$	$\frac{3}{7}$	$\frac{-3}{7}$	$\frac{4}{9}$	$\frac{-4}{9}$	$\frac{5}{6}$	$\frac{-5}{6}$	$\cdots$
	$\frac{1}{6}$	$\frac{-1}{6}$	$\frac{2}{11}$	$\frac{-2}{11}$	$\frac{3}{8}$	$\frac{-3}{8}$	$\frac{4}{11}$	$\frac{-4}{11}$	$\frac{5}{7}$	$\frac{-5}{7}$	$\cdots$
	$\frac{1}{7}$	$\frac{-1}{7}$	$\frac{2}{13}$	$\frac{-2}{13}$	$\frac{3}{10}$	$\frac{-3}{10}$	$\frac{4}{13}$	$\frac{-4}{13}$	$\frac{5}{8}$	$\frac{-5}{8}$	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Next, draw an infinite path in this array, beginning at $\frac{0}{1}$ and snaking back and forth as indicated below. Every rational number is on this path.

0	1	-1	2	-2	3	-3	4	-4	5	-5	$\cdots$
$\frac{0}{1}$	$\frac{1}{1}$	$\frac{-1}{1}$	$\frac{2}{1}$	$\frac{-2}{1}$	$\frac{3}{1}$	$\frac{-3}{1}$	$\frac{4}{1}$	$\frac{-4}{1}$	$\frac{5}{1}$	$\frac{-5}{1}$	$\cdots$
	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{2}{3}$	$\frac{-2}{3}$	$\frac{3}{2}$	$\frac{-3}{2}$	$\frac{4}{3}$	$\frac{-4}{3}$	$\frac{5}{2}$	$\frac{-5}{2}$	$\cdots$
	$\frac{1}{3}$	$\frac{-1}{3}$	$\frac{2}{5}$	$\frac{-2}{5}$	$\frac{3}{4}$	$\frac{-3}{4}$	$\frac{4}{5}$	$\frac{-4}{5}$	$\frac{5}{3}$	$\frac{-5}{3}$	$\cdots$
	$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{2}{7}$	$\frac{-2}{7}$	$\frac{3}{5}$	$\frac{-3}{5}$	$\frac{4}{7}$	$\frac{-4}{7}$	$\frac{5}{4}$	$\frac{-5}{4}$	$\cdots$
	$\frac{1}{5}$	$\frac{-1}{5}$	$\frac{2}{9}$	$\frac{-2}{9}$	$\frac{3}{7}$	$\frac{-3}{7}$	$\frac{4}{9}$	$\frac{-4}{9}$	$\frac{5}{6}$	$\frac{-5}{6}$	$\cdots$
	$\frac{1}{6}$	$\frac{-1}{6}$	$\frac{2}{11}$	$\frac{-2}{11}$	$\frac{3}{8}$	$\frac{-3}{8}$	$\frac{4}{11}$	$\frac{-4}{11}$	$\frac{5}{7}$	$\frac{-5}{7}$	$\cdots$
	$\frac{1}{7}$	$\frac{-1}{7}$	$\frac{2}{13}$	$\frac{-2}{13}$	$\frac{3}{10}$	$\frac{-3}{10}$	$\frac{4}{13}$	$\frac{-4}{13}$	$\frac{5}{8}$	$\frac{-5}{8}$	$\cdots$

Following this path, we get an infinite list of all rational numbers:

$$0,\ 1,\ \frac{1}{2},\ -\frac{1}{2},\ -1,\ 2,\ \frac{2}{3},\ \frac{2}{5},\ -\frac{1}{3},\ \frac{1}{3},\ \frac{1}{4},\ -\frac{1}{4},\ \frac{2}{7},\ -\frac{2}{7},\ -\frac{2}{5},\ -\frac{2}{3},\ -\frac{2}{3},\ -2,\ 3,\ \frac{3}{2},\ \ldots$$

By Theorem 19.3, it follows that $\mathbb{Q}$ is countably infinite, that is, $|\mathbb{Q}| = |\mathbb{N}|$. $\square$

It is also true that the Cartesian product of two countably infinite sets is itself countably infinite, as our next theorem states.

Theorem 19.5. *If A and B are both countably infinite, then so is $A \times B$.*

Proof. Suppose A and B are both countably infinite. By Theorem 19.3, we know we can write A and B in list form as

$$A = \{a_1, a_2, a_3, a_4, \ldots\},$$
$$B = \{b_1, b_2, b_3, b_4, \ldots\}.$$

Figure 19.2 shows how to form an infinite path winding through all of $A \times B$. Therefore $A \times B$ can be written in list form, so it is countably infinite. $\square$

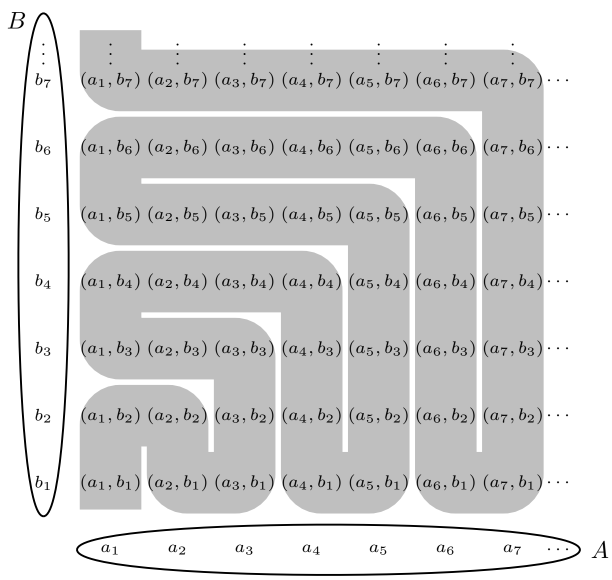


Fig. 19.2 A product of two countably infinite sets is countably infinite

As an example of a consequence of this theorem, notice that since $\mathbb{Q}$ is countably infinite, the set $\mathbb{Q} \times \mathbb{Q}$ is also countably infinite.

Recall that the word "corollary" means a result that follows easily from some other result. We have the following corollary of Theorem 19.5.

Corollary 19.1. *Given n countably infinite sets $A_1, A_2, A_3, \ldots, A_n$, with $n \geq 2$, the Cartesian product $A_1 \times A_2 \times A_3 \times \cdots \times A_n$ is also countably infinite.*

Proof. The proof is by induction on n. For the basis step, notice that when $n = 2$ the statement asserts that for countably infinite sets A_1 and A_2, the product $A_1 \times A_2$ is countably infinite, and this is true by Theorem 19.5.

Assume that for $k \geq 2$, any product $A_1 \times A_2 \times A_3 \times \cdots \times A_k$ of countably infinite sets is countably infinite. Consider a product $A_1 \times A_2 \times A_3 \times \cdots \times A_{k+1}$ of $k+1$ countably infinite sets. It is easily confirmed that the function

$$
\begin{aligned}
f : A_1 \times A_2 \times A_3 \times \cdots \times A_k \times A_{k+1} &\longrightarrow (A_1 \times A_2 \times A_3 \times \cdots \times A_k) \times A_{k+1} \\
f(x_1, x_2, \ldots, x_k, x_{k+1}) &= \big((x_1, x_2, \ldots, x_k),\ x_{k+1}\big)
\end{aligned}
$$

is bijective, so $|A_1 \times A_2 \times A_3 \times \cdots \times A_k \times A_{k+1}| = |(A_1 \times A_2 \times A_3 \times \cdots \times A_k) \times A_{k+1}|$. By the induction hypothesis, $(A_1 \times A_2 \times A_3 \times \cdots \times A_k) \times A_{k+1}$ is a product of two countably infinite sets, so it is countably infinite by Theorem 19.5. As noted above, $A_1 \times A_2 \times A_3 \times \cdots \times A_k \times A_{k+1}$ has the same cardinality, so it too is countably infinite. $\square$

Theorem 19.6. *If A and B are countably infinite, then $A \cup B$ is countably infinite.*

Proof. Suppose A and B are both countably infinite. By Theorem 19.3, we know we can write A and B in list form as

$$
\begin{aligned}
A &= \big\{a_1, a_2, a_3, a_4, \ldots\big\}, \\
B &= \big\{b_1, b_2, b_3, b_4, \ldots\big\}.
\end{aligned}
$$

We can "shuffle" A and B into one infinite list for $A \cup B$ as follows.

$$
A \cup B = \big\{a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, \ldots\big\}.
$$

(We agree not to list an element twice if it belongs to both A and B.) Therefore, by Theorem 19.3, it follows that $A \cup B$ is countably infinite. $\square$

Exercises for Section 19.2

1. Prove that the set $A = \{\ln(n) : n \in \mathbb{N}\} \subseteq \mathbb{R}$ is countably infinite.

2. Prove that the set $A = \{(m, n) \in \mathbb{N} \times \mathbb{N} : m \le n\}$ is countably infinite.

3. Prove that the set $A = \{(5n, -3n) : n \in \mathbb{Z}\}$ is countably infinite.

4. Prove that the set of all irrational numbers is uncountable. (Suggestion: Consider proof by contradiction using Theorems 19.4 and 19.6.)

5. Prove or disprove: There exists a countably infinite subset of the set of irrational numbers.

6. Prove or disprove: There exists a bijective function $f : \mathbb{Q} \to \mathbb{R}$.

7. Prove or disprove: The set $\mathbb{Q}^{100}$ is countably infinite.

8. Prove or disprove: The set $\mathbb{Z} \times \mathbb{Q}$ is countably infinite.

9. Prove or disprove: The set $\{0, 1\} \times \mathbb{N}$ is countably infinite.

10. Prove or disprove: The set $A = \left\{\frac{\sqrt{2}}{n} : n \in \mathbb{N}\right\}$ countably infinite.

11. Describe a partition of $\mathbb{N}$ that divides $\mathbb{N}$ into eight countably infinite subsets.

12. Describe a partition of $\mathbb{N}$ that divides $\mathbb{N}$ into $\aleph_0$ countably infinite subsets.

13. Prove or disprove: If $A = \{X \subseteq \mathbb{N} : X \text{ is finite}\}$, then $|A| = \aleph_0$.

14. Suppose $A = \{(m, n) \in \mathbb{N} \times \mathbb{R} : n = \pi m\}$. Is it true that $|\mathbb{N}| = |A|$?

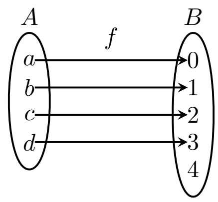
15. Theorem 19.5 implies that $\mathbb{N} \times \mathbb{N}$ is countably infinite. Construct an alternate proof of this fact by showing that the function $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined as $\varphi(m, n) = 2^{n-1}(2m - 1)$ is bijective.

19.3 Comparing Cardinalities

At this point we know that there are at least two different kinds of infinity. On one hand, there are countably infinite sets such as $\mathbb{N}$, of cardinality $\aleph_0$. Then there is the uncountable set $\mathbb{R}$. Are there other kinds of infinity beyond these two kinds? The answer is "yes," but to see why we first need to introduce some new definitions and theorems.

Our first task will be to formulate a definition for what we mean by $|A| < |B|$. Of course if A and B are finite we know exactly what this means: $|A| < |B|$ means that when the elements of A and B are counted, A is found to have fewer elements than B. But this process breaks down if A or B is infinite, for then the elements can't be counted.

The language of functions helps us overcome this difficulty. Notice that for finite sets A and B it is intuitively clear that $|A| < |B|$ if and only if there exists an injective function $f : A \to B$ but there are no surjective functions $f : A \to B$. The following diagram illustrates this:



We will use this idea to define what is meant by $|A| < |B|$ and $|A| \leq |B|$. For emphasis, the following definition also restates what is meant by $|A| = |B|$.

Definition 19.4. Suppose A and B are sets.

- $|A| = |B|$ means there is a bijection $A \to B$.
- $|A| < |B|$ means there is an injection $A \to B$, but no surjection $A \to B$.
- $|A| \leq |B|$ means $|A| < |B|$ or $|A| = |B|$.

For example, consider $\mathbb{N}$ and $\mathbb{R}$. The function $f : \mathbb{N} \to \mathbb{R}$ defined as $f(n) = n$ is clearly injective, but it is not surjective because given the element $\frac{1}{2} \in \mathbb{R}$, we have $f(n) \neq \frac{1}{2}$ for every $n \in \mathbb{N}$. In fact, Theorem 19.2 of Section 19.1 asserts that there is no surjection $\mathbb{N} \to \mathbb{R}$. Definition 19.4 yields

$$|\mathbb{N}| < |\mathbb{R}| \quad \text{or, said differently,} \quad \aleph_0 < |\mathbb{R}|. \tag{19.1}$$

Is there a set X for which $|\mathbb{R}| < |X|$? The next theorem implies $|\mathbb{R}| < |\mathscr{P}(\mathbb{R})|$, so the answer is "yes." (Recall that $\mathscr{P}(A)$ denotes the power set of A.)

Theorem 19.7. *If A is any set, then $|A| < |\mathscr{P}(A)|$.*

Proof. Before beginning the proof, we remark that this statement is obvious if A is finite, for then $|A| < 2^{|A|} = |\mathscr{P}(A)|$. But our proof must apply to *all* sets A, both finite and infinite, so it must use Definition 19.4.

We prove the theorem with direct proof. Let A be an arbitrary set. According to Definition 19.4, to prove $|A| < |\mathscr{P}(A)|$ we must show that there is an injection $f : A \to \mathscr{P}(A)$, but no surjection $f : A \to \mathscr{P}(A)$.

To see that there is an injection $f : A \to \mathscr{P}(A)$, define f by the rule $f(x) = \{x\}$. In words, f sends any element x of A to the one-element set $\{x\} \in \mathscr{P}(A)$. Then $f : A \to \mathscr{P}(A)$ is injective, as follows. Suppose $f(x) = f(y)$. Then $\{x\} = \{y\}$. Consequently, $x = y$, so it follows that $x = y$. Thus f is injective.

Next we need to show that there exists no surjection $f : A \to \mathscr{P}(A)$. Suppose for the sake of contradiction that there does exist a surjection $f : A \to \mathscr{P}(A)$. Notice that for any element $x \in A$, we have $f(x) \in \mathscr{P}(A)$, so $f(x)$ is a subset of A. Thus f is a function that sends elements of A to subsets of A. It follows that for any $x \in A$, either x is an element of the subset $f(x)$ or it is not. Using this idea, define the following subset B of A:

$$B = \{x \in A : x \notin f(x)\} \subseteq A.$$

Because $B \subseteq A$ we have $B \in \mathscr{P}(A)$, and since f is surjective there is an $a \in A$ for which $f(a) = B$. Now, either $a \in B$ or $a \notin B$. We will consider these two cases separately, and show that each leads to a contradiction.

Case 1. If $a \in B$, then the definition of B implies $a \notin f(a)$, and since $f(a) = B$ we have $a \notin B$, which is a contradiction.
Case 2. If $a \notin B$, then the definition of B implies $a \in f(a)$, and since $f(a) = B$ we have $a \in B$, again a contradiction.

Since the assumption that there is a surjection $f : A \to \mathscr{P}(A)$ leads to a contradiction, we conclude that there are no such surjective functions.

In conclusion, we have seen that there exists an injection $A \to \mathscr{P}(A)$ but no surjection $A \to \mathscr{P}(A)$, so Definition 19.4 implies that $|A| < |\mathscr{P}(A)|$. □

Beginning with the set $A = \mathbb{N}$ and applying Theorem 19.7 over and over again, we get the following chain of infinite cardinalities.

$$\aleph_0 \;=\; |\mathbb{N}| \;<\; |\mathscr{P}(\mathbb{N})| \;<\; |\mathscr{P}(\mathscr{P}(\mathbb{N}))| \;<\; |\mathscr{P}(\mathscr{P}(\mathscr{P}(\mathbb{N})))| \;<\; \cdots \tag{19.2}$$

So we have an infinite sequence of different types of infinity, starting with $\aleph_0$ and becoming ever larger. The set $\mathbb{N}$ is countable, and all the sets $\mathscr{P}(\mathbb{N})$, $\mathscr{P}(\mathscr{P}(\mathbb{N}))$, etc., are uncountable. It can be proved that $|\mathscr{P}(\mathbb{N})| = |\mathbb{R}|$. Thus $|\mathbb{N}|$ and $|\mathbb{R}|$ are the first two entries in the chain (19.2) above. They are just two relatively tame infinities in a long list of other wild and exotic infinities.

Unless you plan on studying advanced set theory or the foundations of mathematics, you are unlikely to ever encounter any types of infinity beyond $\aleph_0$ and $|\mathbb{R}|$. Still you may need to distinguish between countably infinite and uncountable sets, so we close with two final theorems for this.

Theorem 19.8. *An infinite subset of a countably infinite set is countably infinite.*

Proof. Suppose A is an infinite subset of the countably infinite set B. Because B is countably infinite, its elements can be written in a list $b_1, b_2, b_3, b_4, \ldots$ Then we can also write A's elements in list form by proceeding through the elements of B, in order, and selecting those that belong to A. Thus A can be written in list form, and since A is infinite, its list will be infinite. Consequently A is countably infinite. □

Theorem 19.9. *If $U \subseteq A$, and U is uncountable, then A is uncountable.*

Proof. Suppose for the sake of contradiction that $U \subseteq A$, and U is uncountable but A is not uncountable. Then since $U \subseteq A$ and U is infinite, then A must be infinite too. Since A is infinite, and not uncountable, it is countably infinite. Then U is an infinite subset of a countably infinite set A, so U is countably infinite by Theorem 19.8. Thus U is both uncountable and countably infinite, a contradiction. □

Theorems 19.8 and 19.9 can be useful when we need to decide whether a set is countably infinite or uncountable. They sometimes allow us to decide its cardinality by comparing it to a set whose cardinality is known.

For example, suppose we want to decide whether or not the set $A = \mathbb{R}^2$ is uncountable. Since the x-axis $U = \big\{(x,0) : x \in \mathbb{R}\big\} \subseteq \mathbb{R}^2$ has the same cardinality as $\mathbb{R}$, it is uncountable. Theorem 19.9 implies that $\mathbb{R}^2$ is uncountable. Other examples can be found in the exercises.

Exercises for Section 19.3

1. Suppose B is an uncountable set and A is a set. Given that there is a surjective function $f : A \to B$, what can be said about the cardinality of A?

2. Prove that the set $\mathbb{C}$ of complex numbers is uncountable.

3. Prove or disprove: If A is uncountable, then $|A| = |\mathbb{R}|$.

4. Prove or disprove: If $A \subseteq B \subseteq C$ and A and C are countably infinite, then B is countably infinite.

5. Prove or disprove: The set $\big\{0,1\big\} \times \mathbb{R}$ is uncountable.

6. Prove or disprove: Every infinite set is a subset of a countably infinite set.

7. Prove or disprove: If $A \subseteq B$ and A is countably infinite and B is uncountable, then $B - A$ is uncountable.

8. Prove or disprove: The set $\big\{(a_1, a_2, a_3, \ldots) : a_i \in \mathbb{Z}\big\}$ of infinite sequences of integers is countably infinite.

9. Prove that if A and B are finite sets with $|A| = |B|$, then any injection $f : A \to B$ is also a surjection. Show this is not necessarily true if A and B are not finite.

10. Prove that if A and B are finite sets with $|A| = |B|$, then any surjection $f : A \to B$ is also an injection. Show this is not necessarily true if A and B are not finite.

19.4 Case Study: Computable Functions

This section uses some of our recent results on cardinality to explore certain theoretical limitations on the problems that computers can solve. Roughly, we will prove that there are more problems than there are algorithms to solve them, and therefore some problems cannot be solved with algorithms.

We will investigate this in the somewhat controlled environment of functions from $\mathbb{N}$ to $\mathbb{N}$. Define a function $f : \mathbb{N} \to \mathbb{N}$ to be **computable** if there exists a procedure that accepts any $n \in \mathbb{N}$ as input, and outputs $f(n)$.

For example, the factorial function $f(n) = n!$ is computable because in Chapter 8 we wrote a procedure `Fac` for which `Fac`$(n) = n!$ (see page 216). Also, the function f for which $f(n)$ is the nth Fibonacci number is a computable function because the procedure `Fib`(n) in the solution of Exercise 8.13 computes it. You can imagine writing a procedure for any function $f(n)$ that applies familiar algebraic operations to n, so such functions are computable. Even an arbitrary piecewise function like

$$g(n) = \begin{cases} 20 - n^2 & \text{if } n \leq 4 \\ 2^n & \text{if } 4 < n < 10 \\ 3 & \text{if } 10 \leq n \end{cases}$$

is computable because below is a procedure for which $g(n) = \texttt{Piece}(n)$.

Procedure Piece(n)

```
begin
    if n ≤ 4 then
        return 20 − n²
    else
        if 4 < n < 10 then
            return 2ⁿ
        else
            return 3
        end
    end
end
```

Given these examples, one might guess that any $f : \mathbb{N} \to \mathbb{N}$ is computable. That is false. In fact, most functions are not computable, and we will spend the remainder of this short section proving it. This involves two steps. In the first step (Proposition 19.1, below) we show that there are uncountably many functions $f : \mathbb{N} \to \mathbb{N}$. Then we show that there are only countably many procedures. This will force us to the conclusion that there are not enough procedures to compute all the functions.

Proposition 19.1. *The set of all functions $f : \mathbb{N} \to \mathbb{N}$ is uncountable.*

Proof. Certainly there are infinitely many functions $f : \mathbb{N} \to \mathbb{N}$. Suppose for the sake of contradiction that this set of functions is countable. Then by Theorem 19.3, these functions can be arranged in an infinite list

$$f_1,\ \ f_2,\ \ f_3,\ \ f_4,\ \ f_5,\ \ldots, \tag{19.3}$$

such that every function $f : \mathbb{N} \to \mathbb{N}$ is somewhere on this list. Imagine the following infinite table that tallies the values of these functions. The table is arranged so that its kth row is $f_k(1),\ \ f_k(2),\ \ f_k(3),\ \ f_k(4), \ldots$, that is, each entry $x_{k\ n}$ is the number $x_{k\ n} = f_k(n)$.

$n:$	1	2	3	4	5	6	7	8	9	$\cdots$
$f_1(n)$	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}	x_{17}	x_{18}	x_{19}	$\cdots$
$f_2(n)$	x_{21}	x_{22}	x_{23}	x_{24}	x_{25}	x_{26}	x_{27}	x_{28}	x_{29}	$\cdots$
$f_3(n)$	x_{31}	x_{32}	x_{33}	x_{34}	x_{35}	x_{36}	x_{37}	x_{38}	x_{39}	$\cdots$
$f_4(n)$	x_{41}	x_{42}	x_{43}	x_{44}	x_{45}	x_{46}	x_{47}	x_{48}	x_{49}	$\cdots$
$f_5(n)$	x_{51}	x_{52}	x_{53}	x_{54}	x_{55}	x_{56}	x_{57}	x_{58}	x_{59}	$\cdots$
$f_6(n)$	x_{61}	x_{62}	x_{63}	x_{64}	x_{65}	x_{66}	x_{67}	x_{68}	x_{69}	$\cdots$
$f_7(n)$	x_{71}	x_{72}	x_{73}	x_{74}	x_{75}	x_{76}	x_{77}	x_{78}	x_{79}	$\cdots$
$f_8(n)$	x_{81}	x_{82}	x_{83}	x_{84}	x_{85}	x_{86}	x_{87}	x_{88}	x_{89}	$\cdots$
$f_9(n)$	x_{91}	x_{92}	x_{93}	x_{94}	x_{95}	x_{96}	x_{97}	x_{98}	x_{99}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

The gray diagonal covers the entries $f_n(n) = x_{n\,n}$. These entries are the key to producing a contradiction, for they lead to a function $f_0 : \mathbb{N} \to \mathbb{N}$ that is not on the list (19.3). Simply define f_0 as

$$f_0(n) = \begin{cases} 1 \text{ if } x_{n\,n} = 0 \\ 0 \text{ if } x_{n\,n} \neq 0. \end{cases}$$

Notice that if f_n is any function on the list (19.3), then $f_n(n) = x_{n\,n} \neq f_0(n)$. Therefore f_n and f_0 are not equal, because they do not agree on the value n. Consequently f_0 is not equal to any of the functions f_n on the list (19.3). This contradicts the assumption we made in the first paragraph of the proof, namely that the list (19.3) contains every function $\mathbb{N} \to \mathbb{N}$.

In summary, the assumption that the set of functions $\mathbb{N} \to \mathbb{N}$ is countable leads to a contradiction, so this set is uncountable. $\square$

Next we will argue that, even though the set of functions $\mathbb{N} \to \mathbb{N}$ is uncountable, the set of *procedures* is *countably infinite*. To fix the discussion, consider procedures that are written in some particular programming language. Programs (procedures) in this language are written using a finite set Σ of computer keyboard symbols. (Here the sigma is a nemonic for "symbol," not "sum.") The set Σ includes the "blank" space character, which we will denote as "␣" (to make it visible). Thus

$$\Sigma = \Big\{ \text{␣}, \text{ a, b, c}, \ldots, \text{ z, A, B}, \ldots, \text{ Z, } 0, 1, 2, \ldots, 9, (,), +, *, \$, \{, \}, /, \ldots \Big\},$$

where the blank character is listed first. Let's say $|\Sigma| = 100$ (a value chosen for simplicity, rather than by actually counting the symbols on a keyboard).

For a fixed positive integer k, the Cartesian power $\Sigma^k = \Sigma \times \Sigma \times \cdots \times \Sigma$ has cardinality 100^k, a number that can be quite large, but it is finite.

We can identify any length-k string of symbols from Σ as an element of Σ^k. For example, the string "for (i=2) to 10 do" appears in Σ^{18} as

$$\Big(\text{f, o, r, \_, (, i, =, 2,) , \_, t, o, \_, 1, 0, \_, d, o} \Big) \in \Sigma^{18}.$$

Certainly Σ^{18} also contains some meaningless nonsense, like the encoding of "uk9*C$aaaA2017hhhh". But any procedure or fragment of a procedure that uses 18 or fewer characters is encoded in Σ^{18}. Likewise, any procedure that is written in k or fewer characters can be regarded an element of Σ^k.

Now consider the set

$$\Upsilon = \bigcup_{k=1}^{\infty} \Sigma^k = \Sigma^1 \cup \Sigma^2 \cup \Sigma^3 \cup \Sigma^4 \cup \cdots$$

Any syntactically correct procedure is an element of this set. Granted, Υ also contains a lot of meaningless nonsense. And it contains some brilliant non-procedures, like Shakepere's *Othello*. But the point is that Υ contains all procedures that can possibly be written.

The interesting thing about Υ is that it is countable. To verify this, we just need to show that its elements can be arranged in an infinite list. This can be done as follows. The first 100 entries of the list are the 100 symbols in Σ^1. Follow this with the $100^2 = 10,000$ elements of Σ^2. Then the list continues with the $100^3 = 1,000,000$ elements of Σ^3, followed by the $100^4 = 100,000,000$ elements of Σ^4, and so on. Therefore Υ is countable.

The set of all possible procedures is a subset of the countably infinite set Υ, so it is countable by Theorem 19.8, which states that an infinite subset of a countable set is countable. We have therefore proved the following result.

Proposition 19.2. *The set of all programs (or procedures) that can be written in a given programming language is countable.*

This means that the set of computable functions—those functions that can be computed with a procedure—is countable, because there are only countably many procedures to compute them. But Proposition 19.3 says that the number of functions $\mathbb{N} \to \mathbb{N}$ is uncountable. Therefore there exist functions that cannot be computed with a procedure. This is our main result.

Theorem 19.10. *There exist functions that are not computable, that is, they cannot be computed by any procedure.*

Solutions for Chapter 19

Section 19.1 Exercises

1. $\mathbb{R}$ and $(0, \infty)$
Observe that the function $f(x) = e^x$ sends $\mathbb{R}$ to $(0, \infty)$. It is injective because $f(x) = f(y)$ implies $e^x = e^y$, and taking ln of both sides gives $x = y$. It is surjective because if $b \in (0, \infty)$, then $f(\ln(b)) = b$. Therefore, because of the bijection $f : \mathbb{R} \to (0, \infty)$, it follows that $|\mathbb{R}| = |(0, \infty)|$.

3. $\mathbb{R}$ and $(0, 1)$
Observe that the function $\frac{1}{\pi} f(x) = \cot^{-1}(x)$ sends $\mathbb{R}$ to $(0, 1)$. It is injective and surjective by elementary trigonometry. Therefore, because of the bijection $f : \mathbb{R} \to (0, 1)$, it follows that $|\mathbb{R}| = |(0, 1)|$.

5. $A = \{3k : k \in \mathbb{Z}\}$ and $B = \{7k : k \in \mathbb{Z}\}$
Observe that the function $f(x) = \frac{7}{3}x$ sends A to B. It is injective because $f(x) = f(y)$ implies $\frac{7}{3}x = \frac{7}{3}y$, and multiplying both sides by $\frac{3}{7}$ gives $x = y$. It is surjective because if $b \in B$, then $b = 7k$ for some integer k. Then $3k \in A$, and $f(3k) = 7k = b$. Therefore, because of the bijection $f : A \to B$, it follows that $|A| = |B|$.

7. $\mathbb{Z}$ and $S = \left\{\dots, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, 16, \dots\right\}$
Observe that the function $f : \mathbb{Z} \to S$ defined as $f(n) = 2^n$ is bijective: It is injective because $f(m) = f(n)$ implies $2^m = 2^n$, and taking $\log_2$ of both sides produces $m = n$. It is surjective because any element b of S has form $b = 2^n$ for some integer n, and therefore $f(n) = 2^n = b$. Because of the bijection $f : \mathbb{Z} \to S$, it follows that $|\mathbb{Z}| = |S|$.

9. $\{0, 1\} \times \mathbb{N}$ and $\mathbb{N}$
Consider the function $f : \{0, 1\} \times \mathbb{N} \to \mathbb{N}$ defined as $f(a, n) = 2n - a$. This is injective because if $f(a, n) = f(b, m)$, then $2n - a = 2m - b$. Now if a were unequal to b, one of a or b would be 0 and the other would be 1, and one side of $2n - a = 2m - b$ would be odd and the other even, a contradiction. Therefore $a = b$. Then $2n - a = 2m - b$ becomes $2n - a = 2m - a$; add a to both sides and divide by 2 to get $m = n$. Thus we have $a = b$ and $m = n$, so $(a, n) = (b, m)$, so f is injective. To see that f is surjective, take any $b \in \mathbb{N}$. If b is even, then $b = 2n$ for some integer n, and $f(0, n) = 2n - 0 = b$. If b is odd, then $b = 2n + 1$ for some integer n. Then $f(1, n + 1) = 2(n + 1) - 1 = 2n + 1 = b$. Therefore f is surjective. Then f is a bijection, so $|\{0, 1\} \times \mathbb{N}| = |\mathbb{N}|$.

11. $[0, 1]$ and $(0, 1)$

Proof. Consider the subset $X = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \subseteq [0, 1]$. Let $f : [0, 1] \to [0, 1)$ be defined as $f(x) = x$ if $x \in [0, 1] - X$ and $f(\frac{1}{n}) = \frac{1}{n+1}$ for any $\frac{1}{n} \in X$. It is easy to check that f is a bijection. Next let $Y = \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\} \subseteq [0, 1)$, and define $g : [0, 1) \to (0, 1)$ as $g(x) = x$ if $x \in [0, 1) - Y$ and $g(1 - \frac{1}{n}) = 1 - \frac{1}{n+1}$ for any $1 - \frac{1}{n} \in Y$. As in the case of f, it is easy to check that g is a bijection. Therefore the composition $g \circ f : [0, 1] \to (0, 1)$ is a bijection. (See Theorem 18.3.) We conclude that $|[0, 1]| = |(0, 1)|$. $\square$

13. $\mathscr{P}(\mathbb{N})$ and $\mathscr{P}(\mathbb{Z})$
Outline: By Exercise 18 of Section 18.2, we have a bijection $f : \mathbb{N} \to \mathbb{Z}$ defined

as $f(n) = \dfrac{(-1)^n(2n-1)+1}{4}$. Now define a function $\Phi : \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{Z})$ as $\Phi(X) = \{f(x) : x \in X\}$. Check that Φ is a bijection.

15. Find a formula for the bijection f in Example 19.2.
Hint: Consider the function f from Exercise 18 of Section 18.2.

Section 19.2 Exercises

1. Prove that the set $A = \{\ln(n) : n \in \mathbb{N}\} \subseteq \mathbb{R}$ is countably infinite.
Just note that its elements can be written in infinite list form as $\ln(1), \ln(2), \ln(3), \cdots$. Thus A is countably infinite.

3. Prove that the set $A = \{(5n, -3n) : n \in \mathbb{Z}\}$ is countably infinite.
Consider the function $f : \mathbb{Z} \to A$ defined as $f(n) = (5n, -3n)$. This is clearly surjective, and it is injective because $f(n) = f(m)$ gives $(5n, -3n) = (5m, -3m)$, so $5n = 5m$, hence $m = n$. Thus, because f is surjective, $|\mathbb{Z}| = |A|$, and $|A| = |\mathbb{Z}| = \aleph_0$. Therefore A is countably infinite.

5. Prove or disprove: There exists a countably infinite subset of the set of irrational numbers.
This is true. Just consider the set consisting of the irrational numbers $\frac{\pi}{1}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \cdots$.

7. Prove or disprove: The set $\mathbb{Q}^{100}$ is countably infinite.
This is true. Note $\mathbb{Q}^{100} = \mathbb{Q} \times \mathbb{Q} \times \cdots \times \mathbb{Q}$ (100 times), and since $\mathbb{Q}$ is countably infinite, it follows from the corollary of Theorem 19.5 that this product is countably infinite.

9. Prove or disprove: The set $\{0,1\} \times \mathbb{N}$ is countably infinite.
This is true. Note that $\{0,1\} \times \mathbb{N}$ can be written in infinite list form as $(0,1), (1,1), (0,2), (1,2), (0,3), (1,3), (0,4), (1,4), \cdots$. Thus the set is countably infinite.

11. Partition $\mathbb{N}$ into 8 countably infinite sets.

For each $i \in \{1,2,3,4,5,6,7,8\}$, let X_i be those natural numbers that are congruent to i modulo 8, that is,

$$\begin{aligned}
X_1 &= \{1, 9, 17, 25, 33, \ldots\} \\
X_2 &= \{2, 10, 18, 26, 34, \ldots\} \\
X_3 &= \{3, 11, 19, 27, 35, \ldots\} \\
X_4 &= \{4, 12, 20, 28, 36, \ldots\} \\
X_5 &= \{5, 13, 21, 29, 37, \ldots\} \\
X_6 &= \{6, 14, 22, 30, 38, \ldots\} \\
X_7 &= \{7, 15, 13, 31, 39, \ldots\} \\
X_8 &= \{8, 16, 24, 32, 40, \ldots\}
\end{aligned}$$

13. If $A = \{X \subset \mathbb{N} : X \text{ is finite}\}$, then $|A| = \aleph_0$.

Proof. This is **true.** To show this we will describe how to arrange the items of A in an infinite list $X_1, X_2, X_3, X_4, \ldots$.

For each natural number n, let p_n be the nth prime number. Thus $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, $p_5 = 11$, and so on. Now consider any element $X \in A$. If $X \neq \emptyset$, then $X = \{n_1, n_2, n_3, ..., n_k\}$, where $k = |X|$ and $n_i \in \mathbb{N}$ for each $1 \leq i \leq k$. Define a function $f : A \to \mathbb{N} \cup \{0\}$ as follows: $f(\{n_1, n_2, n_3, ..., n_k\}) = p_{n_1} p_{n_2} \cdots p_{n_k}$. For example, $f(\{1, 2, 3\}) = p_1 p_2 p_3 = 2 \cdot 3 \cdot 5 = 30$, and $f(\{3, 5\}) = p_3 p_5 = 5 \cdot 11 = 55$, etc. Also, we should not forget that $\emptyset \in A$, and we define $f(\emptyset) = 0$.

Note that $f : A \to \mathbb{N} \cup \{0\}$ is an injection: Let $X = \{n_1, n_2, n_3, ..., n_k\}$ and put $Y = \{m_1, m_2, m_3, ..., m_\ell\}$, and $X \neq Y$. Then there is an integer a that belongs to one of X or Y but not the other. Then the prime factorization of one of the numbers $f(X)$ and $f(Y)$ uses the prime number p_a but the prime factorization of the other does not use p_a. It follows that $f(X) \neq f(Y)$ by the fundamental theorem of arithmetic. Thus f is injective.

So each set $X \in A$ is associated with an integer $f(X) \geq 0$, and no two different sets are associated with the same number. Thus we can list the elements in $X \in A$ in increasing order of the numbers $f(X)$. The list begins as

$$\emptyset, \ \{1\}, \ \{2\}, \ \{3\}, \ \{1,2\}, \ \{4\}, \ \{1,3\}, \ \{5\}, \ \{6\}, \ \{1,4\}, \ \{2,3\}, \ \{7\}, \ldots$$

It follows that A is countably infinite. □

15. Hint: Use the fundamental theorem of arithmetic.

Section 19.3 Exercises

1. Suppose B is an uncountable set and A is a set. Given that there is a surjective function $f : A \to B$, what can be said about the cardinality of A?

The set A must be uncountable, as follows. For each $b \in B$, let a_b be an element of A for which $f(a_b) = b$. (Such an element must exist because f is surjective.) Now form the set $U = \{a_b : b \in B\}$. Then the function $f : U \to B$ is bijective, by construction. Then since B is uncountable, so is U. Therefore U is an uncountable subset of A, so A is uncountable by Theorem 19.9.

3. Prove or disprove: If A is uncountable, then $|A| = |\mathbb{R}|$.

This is false. Let $A = \mathscr{P}(\mathbb{R})$. Then A is uncountable, and by Theorem 19.7, $|\mathbb{R}| < |\mathscr{P}(\mathbb{R})| = |A|$.

5. Prove or disprove: The set $\{0, 1\} \times \mathbb{R}$ is uncountable.

This is true. To see why, first note that the function $f : \mathbb{R} \to \{0\} \times \mathbb{R}$ defined as $f(x) = (0, x)$ is a bijection. Thus $|\mathbb{R}| = |\{0\} \times \mathbb{R}|$, and since $\mathbb{R}$ is uncountable, so is $\{0\} \times \mathbb{R}$. Then $\{0\} \times \mathbb{R}$ is an uncountable subset of the set $\{0, 1\} \times \mathbb{R}$, so $\{0, 1\} \times \mathbb{R}$ is uncountable by Theorem 19.9.

7. Prove or disprove: If $A \subseteq B$ and A is countably infinite and B is uncountable, then $B - A$ is uncountable.

This is true. To see why, suppose to the contrary that $B - A$ is countably infinite. Then $B = A \cup (B - A)$ is a union of countably infinite sets, and thus countable, by Theorem 19.6. This contradicts the fact that B is uncountable.