Chapter 15

Mathematical Induction

This chapter explains a powerful proof technique called **mathematical induction** (or just **induction** for short). To motivate it, let's first examine the kinds of statements that induction is used to prove. Consider the following statement.

Conjecture. The sum of the first n odd natural numbers equals n^2.

The table below illustrates what this conjecture says. Each row starts with a natural number n, followed by the sum of the first n odd natural numbers, then n^2.

n	sum of the first n odd natural numbers	n^2
1	$1 =$	1
2	$1 + 3 =$	4
3	$1 + 3 + 5 =$	9
4	$1 + 3 + 5 + 7 =$	16
5	$1 + 3 + 5 + 7 + 9 =$	25
$\vdots$	$\vdots$	$\vdots$
n	$1 + 3 + 5 + 7 + 9 + 11 + \cdots + (2n - 1) =$	n^2
$\vdots$	$\vdots$	$\vdots$

Note that in the first five lines of the table, the sum of the first n odd numbers really does add up to n^2. Notice also that these first five lines indicate that the nth odd natural number (the last number in each sum) is $2n - 1$. (For instance, when $n = 2$, the second odd natural number is $2 \cdot 2 - 1 = 3$; when $n = 3$, the third odd natural number is $2 \cdot 3 - 1 = 5$, etc.)

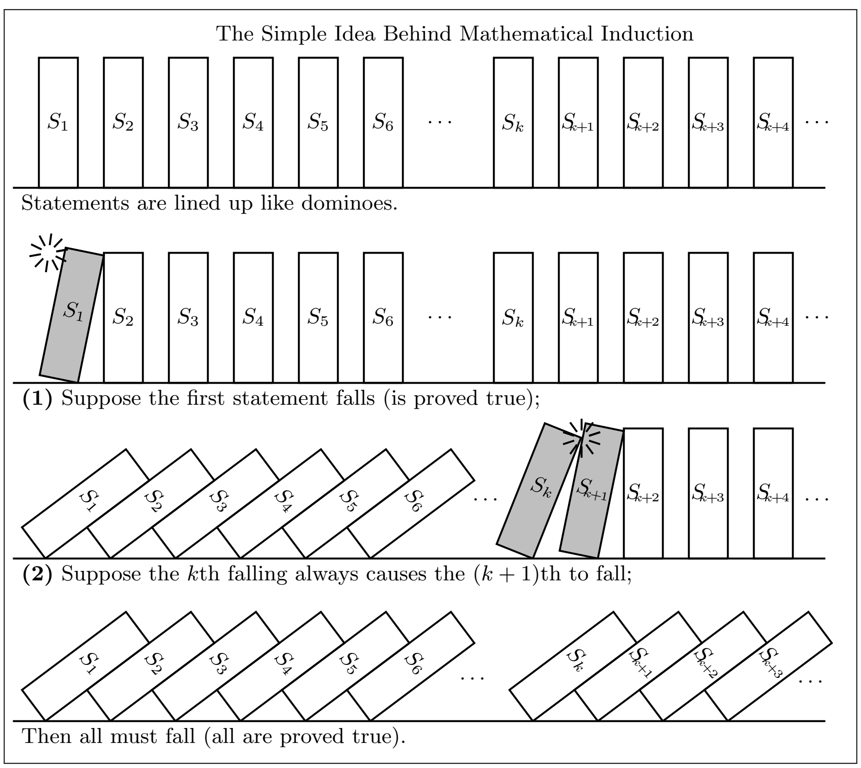
The table raises a question. Does the sum $1 + 3 + 5 + 7 + \cdots + (2n - 1)$ really always equal n^2? In other words, is the conjecture true?

Let's rephrase this. For each natural number n (i.e., for each line of the table), we have a statement S_n, as follows:

$$
\begin{aligned}
&S_1 : 1 = 1^2\\
&S_2 : 1 + 3 = 2^2\\
&S_3 : 1 + 3 + 5 = 3^2\\
&S_4 : 1 + 3 + 5 + 7 = 4^2\\
&\quad\vdots\\
&S_n : 1 + 3 + 5 + 7 + \cdots + (2n-1) = n^2\\
&\quad\vdots
\end{aligned}
$$

Our question is: Are all of these statements true?

Mathematical induction answers just this kind of question, where we have an infinite list of statements $S_1, S_2, S_3, \ldots$ that we want to prove true. The method is simple. To visualize it, think of the statements as dominoes, lined up in a row. Suppose you can prove the first statement S_1, and symbolize this as domino S_1 being knocked down. Also, say you can prove that any statement S_k being true (falling) forces the next statement S_{k+1} to be true (to fall). Then S_1 falls, knocking down S_2. Next S_2 falls, knocking down S_3, then S_3 knocks down S_4, and so on. The inescapable conclusion is that all the statements are knocked down (proved true).



15.1 Proof by Induction

This domino analogy motivates an outline for our next major proof technique: *proof by mathematical induction.*

Outline for Proof by Induction

Proposition The statements $S_1, S_2, S_3, S_4, \ldots$ are all true.

Proof. (Induction)

(1) Prove that the first statement S_1 is true.
(2) Given any integer $k \geq 1$, prove that the statement $S_k \Rightarrow S_{k+1}$ is true.
It follows by mathematical induction that every S_n is true. □

In this setup, the first step (1) is called the **basis step**. Because S_1 is usually a very simple statement, the basis step is often quite easy to do. The second step (2) is called the **inductive step**. In the inductive step direct proof is most often used to prove $S_k \Rightarrow S_{k+1}$, so this step is usually carried out by assuming S_k is true and showing this forces S_{k+1} to be true. The assumption that S_k is true is called the **inductive hypothesis**.

Now let's apply this technique to our original conjecture that the sum of the first n odd natural numbers equals n^2. Our goal is to show that for each $n \in \mathbb{N}$, the statement $S_n : 1+3+5+7+\cdots+(2n-1) = n^2$ is true. Before getting started, observe that S_k is obtained from S_n by plugging k in for n. Thus S_k is the statement $S_k : 1+3+5+7+\cdots+(2k-1) = k^2$. Also, we get S_{k+1} by plugging in $k+1$ for n, so that $S_{k+1} : 1+3+5+7+\cdots+(2(k+1)-1) = (k+1)^2$.

Proposition If $n \in \mathbb{N}$, then $1+3+5+7+\cdots+(2n-1) = n^2$.

Proof. We will prove this with mathematical induction.

(1) Observe that if $n = 1$, this statement is $1 = 1^2$, which is obviously true.
(2) We must now prove $S_k \Rightarrow S_{k+1}$ for any $k \geq 1$. That is, we must show that if $1+3+5+7+\cdots+(2k-1) = k^2$, then $1+3+5+7+\cdots+(2(k+1)-1) = (k+1)^2$. We use direct proof. Suppose $1+3+5+7+\cdots+(2k-1) = k^2$. Then

$$\begin{aligned}
1+3+5+7+\cdots\cdots\cdots\cdots\cdots+(2(k+1)-1) &= \\
1+3+5+7+\cdots+\ (2k-1)\ +(2(k+1)-1) &= \\
\big(1+3+5+7+\cdots+(2k-1)\big)+(2(k+1)-1) &= \\
k^2 \quad +(2(k+1)-1) &= k^2+2k+1 \\
&= (k+1)^2.
\end{aligned}$$

Thus $1+3+5+7+\cdots+(2(k+1)-1) = (k+1)^2$. This proves that $S_k \Rightarrow S_{k+1}$. It follows by induction that $1+3+5+7+\cdots+(2n-1) = n^2$ for every $n \in \mathbb{N}$. □

In induction proofs it is usually the case that the first statement S_1 is indexed by the natural number 1, but this need not always be so. Depending on the problem, the first statement could be S_0, or S_m for any other integer m.

In the next example, the statements are $S_0, S_1, S_2, S_3, \dots$ The same outline is used, except that the basis step verifies S_0, not S_1. We will also have occasion to use the Binomial Theorem (Theorem 6.6 on page 133) in expanding $(k+1)^5 = \binom{5}{0}k^5 1^0 + \binom{5}{1}k^4 1^1 + \binom{5}{2}k^3 1^2 + \binom{5}{3}k^2 1^3 + \binom{5}{4}k^1 1^4 + \binom{5}{5}k^0 1^5$. This simplifies to $(k+1)^5 = k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1$ (with coefficients from the fifth row of Pascal's triangle).

Proposition. If n is a non-negative integer, then $5 \mid (n^5 - n)$.

Proof. We will prove this with mathematical induction. Observe that the first non-negative integer is 0, so the basis step involves $n = 0$.

(1) If $n = 0$, this statement is $5 \mid (0^5 - 0)$ or $5 \mid 0$, which is obviously true.

(2) Let $k \geq 0$. We need to prove that if $5 \mid (k^5 - k)$, then $5 \mid ((k+1)^5 - (k+1))$. We use direct proof. Suppose $5 \mid (k^5 - k)$. Thus $k^5 - k = 5a$ for some $a \in \mathbb{Z}$. Observe that

$$\begin{aligned}
(k+1)^5 - (k+1) &= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1 \\
&= (k^5 - k) + 5k^4 + 10k^3 + 10k^2 + 5k \\
&= 5a + 5k^4 + 10k^3 + 10k^2 + 5k \\
&= 5\big(a + k^4 + 2k^3 + 2k^2 + k\big).
\end{aligned}$$

This shows $(k+1)^5 - (k+1)$ is an integer multiple of 5, so $5 \mid ((k+1)^5 - (k+1))$. We have now shown that $5 \mid (k^5 - k)$ implies $5 \mid ((k+1)^5 - (k+1))$.

It follows by induction that $5 \mid (n^5 - n)$ for all non-negative integers n. □

As noted, induction is used to prove statements of the form $\forall\, n \in \mathbb{N}, S_n$. But notice the outline does *not* work for statements of form $\forall\, n \in \mathbb{Z}, S_n$ (where n is in $\mathbb{Z}$, not $\mathbb{N}$). The reason is that if you are trying to prove $\forall\, n \in \mathbb{Z}, S_n$ by induction, and you've shown S_1 is true and $S_k \Rightarrow S_{k+1}$, then it only follows from this that S_n is true for $n \geq 1$. You haven't proved that any of the statements $S_0, S_{-1}, S_{-2}, \dots$ are true. If you ever want to prove $\forall\, n \in \mathbb{Z}, S_n$ by induction, you have to show that some S_a is true and $S_k \Rightarrow S_{k+1}$ **and** $S_k \Rightarrow S_{k-1}$.

Unfortunately, the term *mathematical induction* is sometimes confused with *inductive reasoning*, that is, the process of reaching the conclusion that something is likely to be true based on prior observations of similar circumstances. Please note that mathematical induction, as introduced here, is a rigorous method that proves statements with absolute certainty.

To round out this section, we present four additional induction proofs.

Proposition. If $n \in \mathbb{Z}$ and $n \geq 0$, then $\displaystyle\sum_{i=0}^{n} i \cdot i! = (n+1)! - 1$.

Proof. We will prove this with mathematical induction.

(1) If $n = 0$, this statement is

$$\sum_{i=0}^{0} i \cdot i! = (0+1)! - 1.$$

Since the left-hand side is $0 \cdot 0! = 0$, and the right-hand side is $1! - 1 = 0$, the equation $\sum_{i=0}^{0} i \cdot i! = (0+1)! - 1$ holds, as both sides are zero.

(2) Consider any integer $k \geq 0$. We must show that S_k implies S_{k+1}. That is, we must show that

$$\sum_{i=0}^{k} i \cdot i! = (k+1)! - 1$$

implies

$$\sum_{i=0}^{k+1} i \cdot i! = ((k+1)+1)! - 1.$$

We use direct proof. Suppose $\displaystyle\sum_{i=0}^{k} i \cdot i! = (k+1)! - 1$. Using this, we get

$$\begin{aligned}
\sum_{i=0}^{k+1} i \cdot i! &= \left(\sum_{i=0}^{k} i \cdot i! \right) + (k+1)(k+1)! \\
&= \Big((k+1)! - 1 \Big) + (k+1)(k+1)! \\
&= (k+1)! + (k+1)(k+1)! - 1 \\
&= \big(1 + (k+1)\big)(k+1)! - 1 \\
&= (k+2)(k+1)! - 1 \\
&= (k+2)! - 1 \\
&= ((k+1)+1)! - 1.
\end{aligned}$$

Therefore $\displaystyle\sum_{i=0}^{k+1} i \cdot i! = ((k+1)+1)! - 1$.

It follows by induction that $\displaystyle\sum_{i=0}^{n} i \cdot i! = (n+1)! - 1$ for every integer $n \geq 0$. $\square$

The next example illustrates a trick that is occasionally useful. You know that you can add equal quantities to both sides of an equation without violating equality. But don't forget that you can add *unequal* quantities to both sides of an *inequality*, as long as the quantity added to the bigger side is bigger than the quantity added to the smaller side. For example, if $x \leq y$ and $a \leq b$, then $x + a \leq y + b$. Similarly, if $x \leq y$ and b is positive, then $x \leq y + b$. This oft-forgotten fact is used in the next proof.

Proposition. The inequality $2^n \leq 2^{n+1} - 2^{n-1} - 1$ holds for each $n \in \mathbb{N}$.

Proof. We will prove this with mathematical induction.

(1) If $n = 1$, this statement is $2^1 \leq 2^{1+1} - 2^{1-1} - 1$, and this simplifies to $2 \leq 4 - 1 - 1$, which is obviously true.

(2) Say $k \geq 1$. We use direct proof to show that $2^k \leq 2^{k+1} - 2^{k-1} - 1$ implies $2^{k+1} \leq 2^{(k+1)+1} - 2^{(k+1)-1} - 1$. Suppose $2^k \leq 2^{k+1} - 2^{k-1} - 1$. Then:

$$\begin{aligned}
2^k &\leq 2^{k+1} - 2^{k-1} - 1 \\
2\big(2^k\big) &\leq 2\big(2^{k+1} - 2^{k-1} - 1\big) && \text{(multiply both sides by 2)} \\
2^{k+1} &\leq 2^{k+2} - 2^k - 2 \\
2^{k+1} &\leq 2^{k+2} - 2^k - 2 + 1 && \text{(add 1 to the bigger side)} \\
2^{k+1} &\leq 2^{k+2} - 2^k - 1 \\
2^{k+1} &\leq 2^{(k+1)+1} - 2^{(k+1)-1} - 1.
\end{aligned}$$

We have now shown that $2^k \leq 2^{k+1} - 2^{k-1} - 1$ being true forces the inequality $2^{k+1} \leq 2^{(k+1)+1} - 2^{(k+1)-1} - 1$ to be true.

It follows by induction that $2^n \leq 2^{n+1} - 2^{n-1} - 1$ for each $n \in \mathbb{N}$. □

Actually, induction was not necessary in the above proposition. Here is an non-inductive approach: Start with the equation $2^n = \frac{1}{2}2^{n+1}$, from which $2^n < \frac{3}{4}2^{n+1}$. From this, $2^n \leq \frac{3}{4}2^{n+1} - 1$ and then $2^n \leq 2^{n+1} - \frac{1}{4}2^{n+1} - 1$, which simplifies as $2^n \leq 2^{n+1} - 2^{n-1} - 1$.

We next prove that if $n \in \mathbb{N}$, then the inequality $(1 + x)^n \geq 1 + nx$ holds for all $x \in \mathbb{R}$ with $x > -1$. Thus we will need to prove that the statement

$$S_n : \ (1 + x)^n \geq 1 + nx \ \text{ for every } x \in \mathbb{R} \text{ with } x > -1$$

is true for every natural number n. This is (only) slightly different from our other examples, which proved statements of the form $\forall n \in \mathbb{N}, P(n)$, where $P(n)$ is a statement about the number n. This time we are proving something of form

$$\forall n \in \mathbb{N},\, P(n, x),$$

where the statement $P(n, x)$ involves not only n, but also a second variable x. (For the record, the inequality $(1 + x)^n \geq 1 + nx$ is known as *Bernoulli's inequality*.)

Proposition. If $n \in \mathbb{N}$, then $(1+x)^n \geq 1+nx$ for all $x \in \mathbb{R}$ with $x > -1$.

Proof. We will prove this with mathematical induction.

(1) For the basis step, notice that when $n = 1$ the statement is $(1+x)^1 \geq 1+1\cdot x$, and this is true because both sides equal $1+x$.

(2) Assume that for some $k \geq 1$, the statement $(1+x)^k \geq 1+kx$ is true for all $x \in \mathbb{R}$ with $x > -1$. From this we need to prove $(1+x)^{k+1} \geq 1+(k+1)x$. Now, $1+x$ is positive because $x > -1$, so we can multiply both sides of $(1+x)^k \geq 1+kx$ by $(1+x)$ without changing the direction of the $\geq$.

$$\begin{aligned}(1+x)^k(1+x) &\geq (1+kx)(1+x)\\ (1+x)^{k+1} &\geq 1+x+kx+kx^2\\ (1+x)^{k+1} &\geq 1+(k+1)x+kx^2\end{aligned}$$

The above term kx^2 is positive, so removing it from the right-hand side will only make that side smaller. Thus we get $(1+x)^{k+1} \geq 1+(k+1)x$. $\square$

Next, an example where the basis step involves more than routine checking. (It will be used later, so it is numbered for reference.)

Proposition 15.1. *Suppose $a_1, a_2, \ldots, a_n$ are n integers, where $n \geq 2$. If p is prime and $p \mid (a_1 \cdot a_2 \cdot a_3 \cdots a_n)$, then $p \mid a_i$ for at least one of the a_i.*

Proof. The proof is induction on n.

(1) The basis step involves $n = 2$. Let p be prime and suppose $p \mid (a_1 a_2)$. We need to show that $p \mid a_1$ or $p \mid a_2$, or equivalently, if $p \nmid a_1$, then $p \mid a_2$. Thus suppose $p \nmid a_1$. Since p is prime, it follows that $\gcd(p, a_1) = 1$. By Proposition 13.1 (on page 307), there are integers k and ℓ for which $1 = pk + a_1\ell$. Multiplying this by a_2 gives

$$a_2 = pka_2 + a_1 a_2 \ell.$$

As we are assuming that p divides $a_1 a_2$, it is clear that p divides the expression $pka_2 + a_1 a_2 \ell$ on the right; hence $p \mid a_2$. We've now proved that if $p \mid (a_1 a_2)$, then $p \mid a_1$ or $p \mid a_2$. This completes the basis step.

(2) Suppose that $k \geq 2$, and $p \mid (a_1 \cdot a_2 \cdots a_k)$ implies then $p \mid a_i$ for some a_i. Now let $p \mid (a_1 \cdot a_2 \cdots a_k \cdot a_{k+1})$. Then $p \mid \big((a_1 \cdot a_2 \cdots a_k) \cdot a_{k+1}\big)$. By what we proved in the basis step, it follows that $p \mid (a_1 \cdot a_2 \cdots a_k)$ or $p \mid a_{k+1}$. This and the inductive hypothesis imply that p divides one of the a_i. $\square$

Please test your understanding now by working a few exercises.

15.2 Proof by Strong Induction

Sometimes in an induction proof it is hard to show that S_k implies S_{k+1}. It may be easier to show some "lower" S_m (with $m < k$) implies S_{k+1}. For such situations there is a slight variant of induction called strong induction. Strong induction works just like regular induction, except that in Step (2) instead of assuming S_k is true and showing this forces S_{k+1} to be true, we assume that *all* the statements $S_1, S_2, \ldots, S_k$ are true and show this forces S_{k+1} to be true. Thus strong induction uses k times as much information as regular induction to force S_{k+1} to be true. The idea is that if the first k dominoes falling always make the $(k+1)$th domino to fall, then all the dominoes must fall. Here is the outline.

Outline for Proof by Strong Induction

Proposition. The statements $S_1, S_2, S_3, S_4, \ldots$ are all true.

Proof. (Strong induction)
(1) Prove the first statement S_1. (Or the first several S_n, if needed.)
(2) Given any integer $k \geq 1$, prove $(S_1 \wedge S_2 \wedge S_3 \wedge \cdots \wedge S_k) \Rightarrow S_{k+1}$. □

This is useful when S_k does not easily imply S_{k+1}. You might be better served by showing some earlier statement (S_{k-1} or S_{k-2} for instance) implies S_k. In strong induction you can use any (or all) of $S_1, S_2, \ldots, S_k$ to prove S_{k+1}.

Here is a classic "first" example of a strong induction proof. The problem is to prove that you can achieve any postage of 8 cents or more, exactly, using only 3¢ and 5¢ stamps. For example, for a postage of 47 cents, you could use nine 3¢ stamps and four 5¢ stamps. Let S_n be the statement S_n: *You can get a postage of exactly n¢ using only 3¢ and 5¢ stamps.* Thus we need to prove all the statements S_8, S_9, S_{10}, $S_{11} \ldots$ are true. In the proof, to show S_{k+1} is true we will need to "go back" two steps from S_k, so the basis step involves verifying the first **two** statements.

Proposition. Any postage of 8¢ or more is possible using 3¢ and 5¢ stamps.

Proof. We will use strong induction.

(1) The proposition is true for a postages of 8 and 9 cents: For 8 cents, use one 3¢ stamp and one 5¢ stamp. For 9 cents, use three 3¢ stamps.
(2) Let $k \geq 9$, and for each $8 \leq m \leq k$, assume a postage of m cents can be obtained exactly with 3¢ and 5¢ stamps. (That is, assume statements $S_8, S_9, \ldots, S_k$ are all true.) We must show that S_{k+1} is true, that is, $(k+1)$-cents postage can be achieved with 3¢ and 5¢ stamps. By assumption, S_{k-2} is true. Thus we can get $(k-2)$-cents postage with 3¢ and 5¢ stamps. Now just add one more 3¢ stamp, and we have $(k-2)+3 = k+1$ cents postage with 3¢ and 5¢ stamps.

This completes the proof by strong induction. □

Our next example proves that $12 \mid (n^4 - n^2)$ for any $n \in \mathbb{N}$. But first, let's see how regular induction is problematic. Regular induction starts by checking $12 \mid (n^4 - n^2)$ for $n = 1$. This reduces to $12 \mid 0$, which is true. Next we assume $12 \mid (k^4 - k^2)$ and try to show that this implies $12 \mid ((k+1)^4 - (k+1)^2)$. Now, $12 \mid (k^4 - k^2)$ means $k^4 - k^2 = 12a$ for some $a \in \mathbb{Z}$. We want to use this to get $(k+1)^4 - (k+1)^2 = 12b$ for some integer b. Working it out,

$$\begin{aligned}(k+1)^4 - (k+1)^2 &= (k^4 + 4k^3 + 6k^2 + 4k + 1) - (k^2 + 2k + 1) \\ &= (k^4 - k^2) + 4k^3 + 6k^2 + 6k \\ &= 12a + 4k^3 + 6k^2 + 6k.\end{aligned}$$

At this point we're stuck because we can't factor out a 12.

Let's try strong induction. Say S_n is the statement $S_n\colon 12 \mid (n^4 - n^2)$. In strong induction, we assume each of $S_1, S_2, \ldots, S_k$ is true, and show that this makes S_{k+1} true. In particular, if S_1 through S_k are true, then S_{k-5} is true, provided $1 \le k - 5 < k$. We will show $S_{k-5} \Rightarrow S_{k+1}$ instead of $S_k \Rightarrow S_{k+1}$. For this to make sense, our basis step must check that $S_1, S_2, S_3, S_4, S_5, S_6$ are all true. Once this is established, $S_{k-5} \Rightarrow S_{k+1}$ will imply that the other S_k are all true. For example, if $k = 6$, then $S_{k-5} \Rightarrow S_{k+1}$ is $S_1 \Rightarrow S_7$, so S_7 is true; for $k = 7$, then $S_{k-5} \Rightarrow S_{k+1}$ is $S_2 \Rightarrow S_8$, so S_8 is true, etc.

Proposition. If $n \in \mathbb{N}$, then $12 \mid (n^4 - n^2)$.

Proof. We will prove this with strong induction.

(1) First note that the statement is true for the first six positive integers:
If $n = 1$, 12 divides $1^4 - 1^2 = 0$. If $n = 4$, 12 divides $4^4 - 4^2 = 240$.
If $n = 2$, 12 divides $2^4 - 2^2 = 12$. If $n = 5$, 12 divides $5^4 - 5^2 = 600$.
If $n = 3$, 12 divides $3^4 - 3^2 = 72$. If $n = 6$, 12 divides $6^4 - 6^2 = 1260$.

(2) For $k \ge 6$, assume $12 \mid (m^4 - m^2)$ for $1 \le m \le k$ (i.e., $S_1, S_2, \ldots, S_k$ are true). We must show S_{k+1} is true, that is, $12 \mid \big((k+1)^4 - (k+1)^2\big)$. Now, S_{k-5} being true means $12 \mid \big((k-5)^4 - (k-5)^2\big)$. To simplify, put $\boxed{k - 5 = \ell}$ so $12 \mid (\ell^4 - \ell^2)$, meaning $\boxed{\ell^4 - \ell^2 = 12a}$ for $a \in \mathbb{Z}$, and $\boxed{k + 1 = \ell + 6}$. Then:

$$\begin{aligned}(k+1)^4 - (k+1)^2 &= (\ell+6)^4 - (\ell+6)^2 \\ &= \ell^4 + 24\ell^3 + 216\ell^2 + 864\ell + 1296 - (\ell^2 + 12\ell + 36) \\ &= (\ell^4 - \ell^2) + 24\ell^3 + 216\ell^2 + 852\ell + 1260 \\ &= 12a + 24\ell^3 + 216\ell^2 + 852\ell + 1260 \\ &= 12\big(a + 2\ell^3 + 18\ell^2 + 71\ell + 105\big).\end{aligned}$$

Because $\big(a + 2\ell^3 + 18\ell^2 + 71\ell + 105\big) \in \mathbb{Z}$, we get $12 \mid \big((k+1)^4 - (k+1)^2\big)$. This shows by strong induction that $12 \mid (n^4 - n^2)$ for every $n \in \mathbb{N}$. $\square$

15.3 Proof by Smallest Counterexample

This section introduces yet another proof technique, called **proof by smallest counterexample**. It is a hybrid of induction and proof by contradiction. It has the nice feature that it leads you straight to a contradiction. It is therefore more "automatic" than the proof by contradiction that was introduced in Chapter 11.

Outline for Proof by Smallest Counterexample

Proposition. The statements $S_1, S_2, S_3, S_4, \ldots$ are all true.

Proof. (Smallest counterexample)
(1) Check that the first statement S_1 is true.
(2) For the sake of contradiction, suppose not every S_n is true.
(3) Let $k > 1$ be the smallest integer for which S_k is **false**.
(4) Then S_{k-1} is true and S_k is false. Use this to get a contradiction. □

Notice that this setup leads you to a point where S_{k-1} is true and S_k is false. It is here, where true and false collide, that you will find a contradiction.

Proposition. If $n \in \mathbb{N}$, then $4 \mid (5^n - 1)$.

Proof. We use proof by smallest counterexample. (We will number the steps to match the outline, but that is not usually done in practice.)

(1) If $n = 1$, then the statement is $4 \mid (5^1 - 1)$, or $4 \mid 4$, which is true.
(2) For sake of contradiction, suppose it's not true that $4 \mid (5^n - 1)$ for all n.
(3) Let $k > 1$ be the smallest integer for which $4 \nmid (5^k - 1)$.
(4) Then $4 \mid (5^{k-1} - 1)$, so there is an integer a for which $5^{k-1} - 1 = 4a$. Then:

$$\begin{aligned} 5^{k-1} - 1 &= 4a \\ 5(5^{k-1} - 1) &= 5 \cdot 4a \\ 5^k - 5 &= 20a \\ 5^k - 1 &= 20a + 4 \\ 5^k - 1 &= 4(5a + 1) \end{aligned}$$

This means $4 \mid (5^k - 1)$, a contradiction, because $4 \nmid (5^k - 1)$ in Step 3. Thus, we were wrong in Step 2 to assume that it is untrue that $4 \mid (5^n - 1)$ for every n. Therefore $4 \mid (5^n - 1)$ is true for every n. □

We next prove the **fundamental theorem of arithmetic**, which says any integer greater than 1 has a unique prime factorization. For example, 12 factors into primes as $12 = 2 \cdot 2 \cdot 3$, and moreover *any* factorization of 12 into primes uses exactly the primes 2, 2 and 3. Our proof combines the techniques of induction, cases, minimum counterexample and the idea of uniqueness of existence outlined at the end of Section 13.3. We dignify this fundamental result with the label of "Theorem."

Theorem 15.1. *(Fundamental Theorem of Arithmetic) Any integer $n > 1$ has a unique prime factorization. That is, if $n = p_1 \cdot p_2 \cdot p_3 \cdots p_k$ and $n = a_1 \cdot a_2 \cdot a_3 \cdots a_\ell$ are two prime factorizations of n, then $k = \ell$, and the primes p_i and a_i are the same, except that they may be in a different order.*

Proof. Suppose $n > 1$. We first use strong induction to show that n has a prime factorization. For the basis step, if $n = 2$, it is prime, so it is already its own prime factorization. Let $n \geq 2$ and assume every integer between 2 and n (inclusive) has a prime factorization. Consider $n + 1$. If it is prime, then it is its own prime factorization. If it is not prime, then it factors as $n+1 = ab$ with $a, b > 1$. Because a and b are both less than $n+1$ they have prime factorizations $a = p_1 \cdot p_2 \cdot p_3 \cdots p_k$ and $b = p'_1 \cdot p'_2 \cdot p'_3 \cdots p'_\ell$. Then

$$n + 1 = ab = (p_1 \cdot p_2 \cdot p_3 \cdots p_k)(p'_1 \cdot p'_2 \cdot p'_3 \cdots p'_\ell)$$

is a prime factorization of $n + 1$. This competes the proof by strong induction that every integer greater than 1 has a prime factorization.

Next we use proof by smallest counterexample to prove that the prime factorization of any $n \geq 2$ is unique. If $n = 2$, then n clearly has only one prime factorization, namely itself. Assume for the sake of contradiction that there is an $n > 2$ that has different prime factorizations $n = p_1 \cdot p_2 \cdot p_3 \cdots p_k$ and $n = a_1 \cdot a_2 \cdot a_3 \cdots a_\ell$. Assume n is the smallest number with this property. From $n = p_1 \cdot p_2 \cdot p_3 \cdots p_k$, we see that $p_1 \mid n$, so $p_1 \mid (a_1 \cdot a_2 \cdot a_3 \cdots a_\ell)$. By Proposition 15.1 (page 333), p_1 divides one of the primes a_i. As a_i is prime, we have $p_1 = a_i$. Dividing $n = p_1 \cdot p_2 \cdot p_3 \cdots p_k = a_1 \cdot a_2 \cdot a_3 \cdots a_\ell$ by $p_1 = a_i$ yields

$$p_2 \cdot p_3 \cdots p_k = a_1 \cdot a_2 \cdot a_3 \cdots a_{i-1} \cdot a_{i+1} \cdots a_\ell.$$

These two factorizations are different, because the two prime factorizations of n were different. (Remember: the primes p_1 and a_i are equal, so the difference appears in the remaining factors, displayed above.) But also the above number $p_2 \cdot p_3 \cdots p_k$ is smaller than n, and this contradicts the fact that n was the smallest number with two different prime factorizations. $\square$

One word of warning about proof by smallest counterexample. In proofs in other textbooks or in mathematical papers, it often happens that the writer doesn't tell you up front that proof by smallest counterexample is being used. Instead, you will have to read through the proof to glean from context that this technique is being used. In fact, the same warning applies to *all* of our proof techniques. If you continue with mathematics, you will gradually gain through experience the ability to analyze a proof and understand exactly what approach is being used when it is not stated explicitly. Frustrations await you, but do not be discouraged by them. Frustration is a natural part of anything that's worth doing.

15.4 Fibonacci Numbers

Leonardo Pisano, now known as Fibonacci, was a mathematician born around 1175 in what is now Italy. His most significant work was a book *Liber Abaci*, which is recognized as a catalyst in medieval Europe's slow transition from Roman numbers to the Hindu-Arabic number system. But he is best known today for a number sequence that he described in his book and that bears his name. The **Fibonacci sequence** is

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233,\ 377,\ldots$$

The numbers that appear in this sequence are called **Fibonacci numbers**. The first two numbers are 1 and 1, and thereafter any entry is the sum of the previous two entries. For example $3+5=8$, and $5+8=13$, etc. We denote the nth term of this sequence as F_n. Thus $F_1=1$, $F_2=1$, $F_3=2$, $F_4=3$, $F_7=13$ and so on. Notice that the Fibonacci Sequence is entirely determined by the rules $F_1=1$, $F_2=1$, and $F_n=F_{n-1}+F_{n-2}$.

We introduce Fibonacci's sequence here partly because it is something everyone should know about, but also because it is a great source of induction problems. This sequence, which appears with surprising frequency in nature, is filled with mysterious patterns and hidden structures. Some of these structures will be revealed to you in the examples and exercises.

We emphasize that the condition $F_n=F_{n-1}+F_{n-2}$ (or equivalently $F_{n+1}=F_n+F_{n-1}$) is the perfect setup for induction. It suggests that we can determine something about F_n by looking at earlier terms of the sequence. In using induction to prove something about the Fibonacci sequence, you should expect to use the equation $F_n=F_{n-1}+F_{n-2}$ somewhere.

For our first example we will prove that $F_{n+1}^2-F_{n+1}F_n-F_n^2=(-1)^n$ for any natural number n. For example, if $n=5$ we have $F_6^2-F_6F_5-F_5^2=8^2-8\cdot 5-5^2=64-40-25=-1=(-1)^5$.

Proposition. The Fibonacci sequence obeys $F_{n+1}^2-F_{n+1}F_n-F_n^2=(-1)^n$.

Proof. We will prove this with mathematical induction.

(1) If $n=1$ we have $F_{n+1}^2-F_{n+1}F_n-F_n^2=F_2^2-F_2F_1-F_1^2=1^2-1\cdot 1-1^2=-1=(-1)^1=(-1)^n$, so indeed $F_{n+1}^2-F_{n+1}F_n-F_n^2=(-1)^n$ is true when $n=1$.

(2) Take any integer $k\geq 1$. We must show that if $F_{k+1}^2-F_{k+1}F_k-F_k^2=(-1)^k$, then $F_{k+2}^2-F_{k+2}F_{k+1}-F_{k+1}^2=(-1)^{k+1}$. We use direct proof. Suppose $F_{k+1}^2-F_{k+1}F_k-F_k^2=(-1)^k$. Now we are going to carefully work out the expression $F_{k+2}^2-F_{k+2}F_{k+1}-F_{k+1}^2$ and show that it really does equal $(-1)^{k+1}$. In so doing, we will use the fact that $F_{k+2}=F_{k+1}+F_k$.

$$\begin{aligned}
F_{k+2}^2 - F_{k+2}F_{k+1} - F_{k+1}^2 &= (F_{k+1} + F_k)^2 - (F_{k+1} + F_k)F_{k+1} - F_{k+1}^2 \\
&= F_{k+1}^2 + 2F_{k+1}F_k + F_k^2 - F_{k+1}^2 - F_kF_{k+1} - F_{k+1}^2 \\
&= -F_{k+1}^2 + F_{k+1}F_k + F_k^2 \\
&= -(F_{k+1}^2 - F_{k+1}F_k - F_k^2) \\
&= -(-1)^k \qquad \text{(inductive hypothesis)} \\
&= (-1)^1(-1)^k \\
&= (-1)^{k+1}
\end{aligned}$$

Therefore $F_{k+2}^2 - F_{k+2}F_{k+1} - F_{k+1}^2 = (-1)^{k+1}$.
It follows by induction that $F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n$ for every $n \in \mathbb{N}$. □

Let's pause for a moment and think about what the result we just proved means. Dividing both sides of $F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n$ by F_n^2 gives

$$\left(\frac{F_{n+1}}{F_n}\right)^2 - \frac{F_{n+1}}{F_n} - 1 = \frac{(-1)^n}{F_n^2}.$$

For large values of n, the right-hand side is very close to zero, and the left-hand side is F_{n+1}/F_n plugged into the polynomial $x^2 - x - 1$. Thus, as n increases, the ratio F_{n+1}/F_n approaches a root of $x^2 - x - 1 = 0$. By the quadratic formula, the roots of $x^2 - x - 1$ are $\frac{1\pm\sqrt{5}}{2}$. As $F_{n+1}/F_n > 1$, this ratio must be approaching the *positive* root $\frac{1+\sqrt{5}}{2}$. Therefore

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}. \tag{15.1}$$

For a quick spot check, note that $F_{13}/F_{12} \approx 1.618025$, while $\frac{1+\sqrt{5}}{2} \approx 1.618033$. Even for the small value $n = 12$, the numbers match to four decimal places.

The number $\Phi = \frac{1+\sqrt{5}}{2}$ is sometimes called the **golden ratio**, and there has been much speculation about its occurrence in nature as well as in classical art and architecture. One theory holds that the Parthenon and the Great Pyramids of Egypt were designed in accordance with this number.

But we are here concerned with things that can be proved. We close by observing how the Fibonacci sequence in many ways resembles a geometric sequence. Recall that a **geometric sequence** with first term a and common ratio r has the form

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ ar^5,\ ar^6,\ ar^7,\ ar^8, \ldots$$

where any term is obtained by multiplying the previous term by r. In general its nth term is $G_n = ar^n$, and $G_{n+1}/G_n = r$. Equation (15.1) tells us that $F_{n+1}/F_n \approx \Phi$. Thus even though it is not a geometric sequence, the Fibonacci sequence tends to behave like a geometric sequence with common ratio Φ, and the further "out" you go, the higher the resemblance.

15.5 Case Study: Proving Recursive Procedures Work

In Section 8.6 (page 216), we devised the following procedure `RFac` for calculating the factorial of an integer n, that is, $\texttt{RFac}(n)$ supposedly returns the value $n!$. This procedure is *recursive*, meaning that within its body there is another call to `RFac`. Although this may seem circular, most high-level programming languages do allow for recursive procedures.

Procedure RFac(n)

```
1 begin
2     if n = 0 then
3         return 1 ............................................ because 0! = 1
4     else
5         return n · RFac(n − 1) ..................... because n! = n · (n − 1)!
6     end
7 end
```

Induction can prove that properly-written recursive procedures are valid, and run correctly when implemented in programming languages that allow for recursion. As an example, we will prove that $\texttt{RFac}(n)$ really does return the correct value of $n!$.

Proposition 15.2. *If n is a non-negative integer, then $\texttt{RFac}(n)$ returns the correct value of $n!$.*

Proof. We will prove this with mathematical induction.

(1) For the base case, suppose $n = 0$. Referring to lines 2 and 3 of `RFac`, we see that $\texttt{RFac}(0)$ returns 1, which is indeed $0!$.

(2) Now take any integer $k \geq 0$. We need to show that if $\texttt{RFac}(k)$ returns $k!$, then $\texttt{RFac}(k+1)$ returns $(k+1)!$.
For this we use direct proof. Thus assume that $\texttt{RFac}(k)$ returns the correct value of $k!$. Now run $\texttt{RFac}(k+1)$. Because $k+1 > 0$, the procedure executes the else clause, and in line 5 it returns the value of

$$(k+1)\cdot\texttt{RFac}\big((k+1)-1\big) = (k+1)\cdot\texttt{RFac}(k).$$

By assumption, $\texttt{RFac}(k)$ in the above line returns the value $k!$, so the above line is $(k+1)\cdot\texttt{RFac}(k) = (k+1)k! = (k+1)!$. Thus $\texttt{RFac}(k+1)$ returns $(n+1)!$.

It follows by induction that $\texttt{RFac}(n)$ returns $n!$ for any integer $n \geq 0$. $\square$

Exercises for Chapter 15

Prove the following statements with either induction, strong induction or proof by smallest counterexample.

1. Prove that $1+2+3+4+\cdots+n=\dfrac{n^2+n}{2}$ for positive integers n.

2. Prove that $1^2+2^2+3^2+4^2+\cdots+n^2=\dfrac{n(n+1)(2n+1)}{6}$ for positive integers n.

3. Prove that $1^3+2^3+3^3+4^3+\cdots+n^3=\dfrac{n^2(n+1)^2}{4}$ for every positive integer n.

4. If $n \in \mathbb{N}$, then $1\cdot 2+2\cdot 3+3\cdot 4+4\cdot 5+\cdots+n(n+1)=\dfrac{n(n+1)(n+2)}{3}$.

5. If $n \in \mathbb{N}$, then $2^1+2^2+2^3+\cdots+2^n=2^{n+1}-2$.

6. Prove that $\displaystyle\sum_{i=1}^{n}(8i-5)=4n^2-n$ for every positive integer n.

7. If $n \in \mathbb{N}$, then $1\cdot 3+2\cdot 4+3\cdot 5+4\cdot 6+\cdots+n(n+2)=\dfrac{n(n+1)(2n+7)}{6}$.

8. If $n \in \mathbb{N}$, then $\dfrac{1}{2!}+\dfrac{2}{3!}+\dfrac{3}{4!}+\cdots+\dfrac{n}{(n+1)!}=1-\dfrac{1}{(n+1)!}$

9. Prove that $24 \mid (5^{2n}-1)$ for every integer $n \geq 0$.

10. Prove that $3 \mid (5^{2n}-1)$ for every integer $n \geq 0$.

11. Prove that $3 \mid (n^3+5n+6)$ for every integer $n \geq 0$.

12. Prove that $9 \mid (4^{3n}+8)$ for every integer $n \geq 0$.

13. Prove that $6 \mid (n^3-n)$ for every integer $n \geq 0$.

14. Suppose $a \in \mathbb{Z}$. Prove that $5 \mid 2^n a$ implies $5 \mid a$ for any $n \in \mathbb{N}$.

15. If $n \in \mathbb{N}$, then $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\dfrac{1}{4\cdot 5}+\cdots+\dfrac{1}{n(n+1)}=1-\dfrac{1}{n+1}$.

16. Prove that that $2^n+1 \leq 3^n$ for every positive integer n.

17. Suppose $A_1, A_2, \ldots A_n$ are sets in some universal set U, and $n \geq 2$. Prove that $\overline{A_1 \cap A_2 \cap \cdots \cap A_n}=\overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n}$.

18. Suppose $A_1, A_2, \ldots A_n$ are sets in some universal set U, and $n \geq 2$. Prove that $\overline{A_1 \cup A_2 \cup \cdots \cup A_n}=\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}$.

19. Prove that $\dfrac{1}{1}+\dfrac{1}{4}+\dfrac{1}{9}+\cdots+\dfrac{1}{n^2} \leq 2-\dfrac{1}{n}$ for every $n \in \mathbb{N}$.

20. Prove that $(1+2+3+\cdots+n)^2=1^3+2^3+3^3+\cdots+n^3$ for every $n \in \mathbb{N}$.

21. If $n \in \mathbb{N}$, then $\dfrac{1}{1}+\dfrac{1}{2}+\dfrac{1}{3}+\dfrac{1}{4}+\dfrac{1}{5}+\cdots+\dfrac{1}{2^n-1}+\dfrac{1}{2^n} \geq 1+\dfrac{n}{2}$.
(Note: This problem asserts that the sum of the first 2^n terms of the harmonic series is at least $1+n/2$. It thus implies that the harmonic series diverges.)

22. If $n \in \mathbb{N}$, then $\left(1-\dfrac{1}{2}\right)\left(1-\dfrac{1}{4}\right)\left(1-\dfrac{1}{8}\right)\left(1-\dfrac{1}{16}\right)\cdots\left(1-\dfrac{1}{2^n}\right) \geq \dfrac{1}{4}+\dfrac{1}{2^{n+1}}$.

23. Use mathematical induction to prove the binomial theorem (Theorem 6.6 on page 133). You may find that you need Equation (6.3) on page 131.

24. Prove that $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$ for each natural number n.

25. Concerning the Fibonacci sequence, prove that $F_1+F_2+F_3+F_4+\ldots+F_n = F_{n+2}-1$.

26. Concerning the Fibonacci sequence, prove that $\sum_{k=1}^{n} F_k^2 = F_nF_{n+1}$.

27. Concerning the Fibonacci sequence, prove that $F_1+F_3+F_5+F_7+\ldots+F_{2n-1} = F_{2n}$.

28. Concerning the Fibonacci sequence, prove that $F_2 + F_4 + F_6 + F_8 + \ldots + F_{2n} = F_{2n+1} - 1$.

29. In this problem $n \in \mathbb{N}$ and F_n is the nth Fibonacci number. Prove that

$$\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \binom{n-3}{3} + \cdots + \binom{0}{n} = F_{n+1}.$$

(For example, $\binom{6}{0} + \binom{5}{1} + \binom{4}{2} + \binom{3}{3} + \binom{2}{4} + \binom{1}{5} + \binom{0}{6} = 1+5+6+1+0+0+0 = 13 = F_{6+1}$.)

30. Here F_n is the nth Fibonacci number. Prove that

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

31. Prove that $\sum_{k=0}^{n} \binom{k}{r} = \binom{n+1}{r+1}$, where $1 \leq r \leq n$.

32. Prove that the number of n-digit binary numbers that have no consecutive 1's is the Fibonacci number F_{n+2}. For example, for $n = 2$ there are three such numbers (00, 01, and 10), and $3 = F_{2+2} = F_4$. Also, for $n = 3$ there are five such numbers (000, 001, 010, 100, 101), and $5 = F_{3+2} = F_5$.

33. Suppose n (infinitely long) straight lines lie on a plane in such a way that no two of the lines are parallel, and no three of the lines intersect at a single point. Show that this arrangement divides the plane into $\frac{n^2+n+2}{2}$ regions.

34. Prove that $3^1 + 3^2 + 3^3 + 3^4 + \cdots + 3^n = \dfrac{3^{n+1} - 3}{2}$ for every $n \in \mathbb{N}$.

35. Prove that if $n, k \in \mathbb{N}$, and n is even and k is odd, then $\binom{n}{k}$ is even.

36. Prove that if $n = 2^k - 1$ for some $k \in \mathbb{N}$, then every entry in the nth row of Pascal's triangle is odd.

37. Prove that if $m, n \in \mathbb{N}$, then $\sum_{k=0}^{n} k\binom{m+k}{m} = n\binom{m+n+1}{m+1} - \binom{m+n+1}{m+2}$.

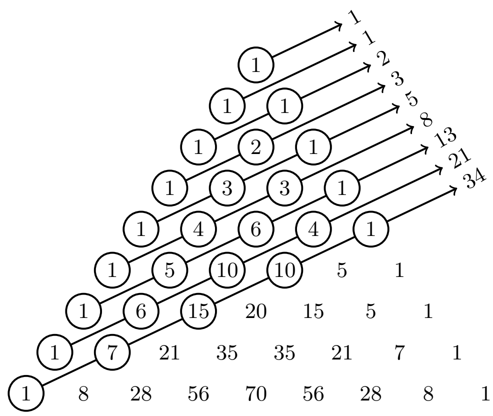
38. Prove that if $n, k \in \mathbb{N}$, then $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$.
(Note that this equality was proved by combinatorial proof in Section 6.10, but here you are asked to prove it by induction.)

39. If n and k are non-negative integers, then $\binom{n+0}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+k}{k} = \binom{n+k+1}{k}$.

40. Prove that $\sum_{k=0}^{p} \binom{m}{k}\binom{n}{p-k} = \binom{m+n}{p}$ for non-negative integers m, n and p.

41. Prove that $\sum_{k=0}^{m} \binom{m}{k}\binom{n}{p+k} = \binom{m+n}{m+p}$ for non-negative integers m, n and p.

42. The indicated diagonals of Pascal's triangle sum to Fibonacci numbers. Prove that this pattern continues forever.



Solutions for Chapter 15

1. Prove that $1+2+3+4+\cdots+n = \frac{n^2+n}{2}$ for every positive integer n.

Proof. We will prove this with mathematical induction.

(1) Observe that if $n=1$, this statement is $1 = \frac{1^2+1}{2}$, which is obviously true.

(2) Consider any integer $k \geq 1$. We must show that S_k implies S_{k+1}. In other words, we must show that if $1+2+3+4+\cdots+k = \frac{k^2+k}{2}$ is true, then

$$1+2+3+4+\cdots+k+(k+1) = \frac{(k+1)^2+(k+1)}{2}$$

is also true. We use direct proof.
Suppose $k \geq 1$ and $1+2+3+4+\cdots+k = \frac{k^2+k}{2}$. Observe that

$$\begin{aligned}
1+2+3+4+\cdots+k+(k+1) &= \\
(1+2+3+4+\cdots+k)+(k+1) &= \\
\frac{k^2+k}{2}+(k+1) &= \frac{k^2+k+2(k+1)}{2} \\
&= \frac{k^2+2k+1\ +\ k+1}{2} \\
&= \frac{(k+1)^2+(k+1)}{2}.
\end{aligned}$$

Therefore we have shown that $1+2+3+4+\cdots+k+(k+1) = \frac{(k+1)^2+(k+1)}{2}$. $\square$

3. Prove that $1^3+2^3+3^3+4^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4}$ for every positive integer n.

Proof. We will prove this with mathematical induction.

(1) When $n=1$ the statement is $1^3 = \frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1$, which is true.

(2) Now assume the statement is true for some integer $n = k \geq 1$, that is assume $1^3+2^3+3^3+4^3+\cdots+k^3 = \frac{k^2(k+1)^2}{4}$. Observe that this implies the statement is true for $n = k+1$.

$$\begin{aligned}
1^3+2^3+3^3+4^3+\cdots+k^3+(k+1)^3 &= \\
(1^3+2^3+3^3+4^3+\cdots+k^3)+(k+1)^3 &= \\
\frac{k^2(k+1)^2}{4}+(k+1)^3 &= \frac{k^2(k+1)^2}{4}+\frac{4(k+1)^3}{4} \\
&= \frac{k^2(k+1)^2+4(k+1)^3}{4} \\
&= \frac{(k+1)^2(k^2+4(k+1)^1)}{4} \\
&= \frac{(k+1)^2(k^2+4k+4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \frac{(k+1)^2((k+1)+1)^2}{4}
\end{aligned}$$

Therefore $1^3+2^3+3^3+4^3+\cdots+k^3+(k+1)^3 = \frac{(k+1)^2((k+1)+1)^2}{4}$, which means the statement is true for $n=k+1$. $\square$

5. If $n \in \mathbb{N}$, then $2^1+2^2+2^3+\cdots+2^n = 2^{n+1}-2$.

Proof. The proof is by mathematical induction.

(1) When $n=1$, this statement is $2^1 = 2^{1+1}-2$, or $2=4-2$, which is true.

(2) Now assume the statement is true for some integer $n=k\geq 1$, that is assume $2^1+2^2+2^3+\cdots+2^k = 2^{k+1}-2$. Observe this implies that the statement is true for $n=k+1$, as follows:

$$\begin{aligned} 2^1+2^2+2^3+\cdots+2^k+2^{k+1} &= \\ (2^1+2^2+2^3+\cdots+2^k)+2^{k+1} &= \\ 2^{k+1}-2+2^{k+1} &= 2\cdot 2^{k+1}-2 \\ &= 2^{k+2}-2 \\ &= 2^{(k+1)+1}-2 \end{aligned}$$

Thus we have $2^1+2^2+2^3+\cdots+2^k+2^{k+1} = 2^{(k+1)+1}-2$, so the statement is true for $n=k+1$.

Thus the result follows by mathematical induction. $\square$

7. If $n \in \mathbb{N}$, then $1\cdot 3+2\cdot 4+3\cdot 5+4\cdot 6+\cdots+n(n+2) = \dfrac{n(n+1)(2n+7)}{6}$.

Proof. The proof is by mathematical induction.

(1) When $n=1$, we have $1\cdot 3 = \frac{1(1+1)(2+7)}{6}$, which is the true statement $3=\frac{18}{6}$.

(2) Now assume the statement is true for some integer $n=k\geq 1$, that is assume $1\cdot 3+2\cdot 4+3\cdot 5+4\cdot 6+\cdots+k(k+2) = \frac{k(k+1)(2k+7)}{6}$. Now observe that

$$\begin{aligned} 1\cdot 3+2\cdot 4+3\cdot 5+4\cdot 6+\cdots+k(k+2)+(k+1)((k+1)+2) &= \\ (1\cdot 3+2\cdot 4+3\cdot 5+4\cdot 6+\cdots+k(k+2))+(k+1)((k+1)+2) &= \\ \frac{k(k+1)(2k+7)}{6}+(k+1)((k+1)+2) &= \\ \frac{k(k+1)(2k+7)}{6}+\frac{6(k+1)(k+3)}{6} &= \\ \frac{k(k+1)(2k+7)+6(k+1)(k+3)}{6} &= \\ \frac{(k+1)(k(2k+7)+6(k+3))}{6} &= \\ \frac{(k+1)(2k^2+13k+18)}{6} &= \\ \frac{(k+1)(k+2)(2k+9)}{6} &= \\ \frac{(k+1)((k+1)+1)(2(k+1)+7)}{6} & \end{aligned}$$

Thus we have $1\cdot 3+2\cdot 4+3\cdot 5+4\cdot 6+\cdots+k(k+2)+(k+1)((k+1)+2) = \frac{(k+1)((k+1)+1)(2(k+1)+7)}{6}$, and this means the statement is true for $n=k+1$.

Thus the result follows by mathematical induction. $\square$

9. Prove that $24 \mid (5^{2n} - 1)$ for every integer $n \geq 0$.

Proof. The proof is by mathematical induction.

(1) For $n = 0$, the statement is $24 \mid (5^{2\cdot 0} - 1)$. This is $24 \mid 0$, which is true.

(2) Now assume the statement is true for some integer $n = k \geq 1$, that is assume $24 \mid (5^{2k} - 1)$. This means $5^{2k} - 1 = 24a$ for some integer a, and from this we get $5^{2k} = 24a + 1$. Now observe that

$$\begin{aligned} 5^{2(k+1)} - 1 &= \\ 5^{2k+2} - 1 &= \\ 5^2 5^{2k} - 1 &= \\ 5^2(24a + 1) - 1 &= \\ 25(24a + 1) - 1 &= \\ 25 \cdot 24a + 25 - 1 &= 24(25a + 1). \end{aligned}$$

This shows $5^{2(k+1)} - 1 = 24(25a + 1)$, which means $24 \mid 5^{2(k+1)} - 1$.

This completes the proof by mathematical induction. □

11. Prove that $3 \mid (n^3 + 5n + 6)$ for every integer $n \geq 0$.

Proof. The proof is by mathematical induction.

(1) When $n = 0$, the statement is $3 \mid (0^3 + 5 \cdot 0 + 6)$, or $3 \mid 6$, which is true.

(2) Now assume the statement is true for some integer $n = k \geq 0$, that is assume $3 \mid (k^3 + 5k + 6)$. This means $k^3 + 5k + 6 = 3a$ for some integer a. We need to show that $3 \mid ((k + 1)^3 + 5(k + 1) + 6)$. Observe that

$$\begin{aligned} (k + 1)^3 + 5(k + 1) + 6 &= k^3 + 3k^2 + 3k + 1 + 5k + 5 + 6 \\ &= (k^3 + 5k + 6) + 3k^2 + 3k + 6 \\ &= 3a + 3k^2 + 3k + 6 \\ &= 3(a + k^2 + k + 2). \end{aligned}$$

Thus we have deduced $(k+1)^3 - (k+1) = 3(a+k^2+k+2)$. Since $a+k^2+k+2$ is an integer, it follows that $3 \mid ((k + 1)^3 + 5(k + 1) + 6)$.

It follows by mathematical induction that $3 \mid (n^3 + 5n + 6)$ for every $n \geq 0$. □

13. Prove that $6 \mid (n^3 - n)$ for every integer $n \geq 0$.

Proof. The proof is by mathematical induction.

(1) When $n = 0$, the statement is $6 \mid (0^3 - 0)$, or $6 \mid 0$, which is true.

(2) Now assume the statement is true for some integer $n = k \geq 0$, that is, assume $6 \mid (k^3 - k)$. This means $k^3 - k = 6a$ for some integer a. We need to show that $6 \mid ((k + 1)^3 - (k + 1))$. Observe that

$$\begin{aligned} (k + 1)^3 - (k + 1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= (k^3 - k) + 3k^2 + 3k \\ &= 6a + 3k^2 + 3k \\ &= 6a + 3k(k + 1). \end{aligned}$$

Thus we have deduced $(k+1)^3-(k+1)=6a+3k(k+1)$. Since one of k or $(k+1)$ must be even, it follows that $k(k+1)$ is even, so $k(k+1)=2b$ for some integer b. Consequently $(k+1)^3-(k+1)=6a+3k(k+1)=6a+3(2b)=6(a+b)$. Since $(k+1)^3-(k+1)=6(a+b)$ it follows that $6 \mid ((k+1)^3-(k+1))$.

Thus the result follows by mathematical induction. □

15. If $n \in \mathbb{N}$, then $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{n(n+1)}=1-\frac{1}{n+1}$.

Proof. The proof is by mathematical induction.

(1) When $n=1$, the statement is $\frac{1}{1(1+1)}=1-\frac{1}{1+1}$, which simplifies to $\frac{1}{2}=\frac{1}{2}$.

(2) Now assume the statement is true for some integer $n=k\geq 1$, that is assume $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{k(k+1)}=1-\frac{1}{k+1}$. Next we show that the statement for $n=k+1$ is true. Observe that

$$\begin{aligned}
\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)((k+1)+1)} &= \\
\left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{k(k+1)}\right)+\frac{1}{(k+1)(k+2)} &= \\
\left(1-\frac{1}{k+1}\right)+\frac{1}{(k+1)(k+2)} &= \\
1-\frac{1}{k+1}+\frac{1}{(k+1)(k+2)} &= \\
1-\frac{k+2}{(k+1)(k+2)}+\frac{1}{(k+1)(k+2)} &= \\
1-\frac{k+1}{(k+1)(k+2)} &= \\
1-\frac{1}{k+2} &= \\
1-\frac{1}{(k+1)+1}&.
\end{aligned}$$

This establishes $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{(k+1)((k+1)+1}=1-\frac{1}{(k+1)+1}$, which is to say that the statement is true for $n=k+1$.

This completes the proof by mathematical induction. □

17. Suppose $A_1, A_2, \ldots A_n$ are sets in some universal set U, and $n\geq 2$. Prove that $\overline{A_1\cap A_2\cap\cdots\cap A_n}=\overline{A_1}\cup\overline{A_2}\cup\cdots\cup\overline{A_n}$.

Proof. The proof is by strong induction.

(1) When $n=2$ the statement is $\overline{A_1\cap A_2}=\overline{A_1}\cup\overline{A_2}$. This is not an entirely

obvious statement, so we have to prove it. Observe that

$$\begin{aligned}
\overline{A_1 \cap A_2} &= \{x : (x \in U) \wedge (x \notin A_1 \cap A_2)\} \quad \text{(definition of complement)} \\
&= \{x : (x \in U) \wedge \neg(x \in A_1 \cap A_2)\} \\
&= \{x : (x \in U) \wedge \neg((x \in A_1) \wedge (x \in A_2))\} \quad \text{(definition of } \cap) \\
&= \{x : (x \in U) \wedge (\neg(x \in A_1) \vee \neg(x \in A_2))\} \quad \text{(DeMorgan)} \\
&= \{x : (x \in U) \wedge ((x \notin A_1) \vee (x \notin A_2))\} \\
&= \{x : (x \in U) \wedge (x \notin A_1) \vee (x \in U) \wedge (x \notin A_2)\} \text{ (distributive prop.)} \\
&= \{x : ((x \in U) \wedge (x \notin A_1))\} \cup \{x : ((x \in U) \wedge (x \notin A_2))\} \text{ (def. of } \cup) \\
&= \overline{A_1} \cup \overline{A_2} \quad \text{(definition of complement)}
\end{aligned}$$

(2) Let $k \geq 2$. Assume the statement is true if it involves k or fewer sets. Then

$$\begin{aligned}
&\overline{A_1 \cap A_2 \cap \cdots \cap A_{k-1} \cap A_k \cap A_{k+1}} = \\
&\overline{A_1 \cap A_2 \cap \cdots \cap A_{k-1} \cap (A_k \cap A_{k+1})} = \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_{k-1}} \cup \overline{A_k \cap A_{k+1}} \\
&\qquad = \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_{k-1}} \cup \overline{A_k} \cup \overline{A_{k+1}}
\end{aligned}$$

Thus the statement is true when it involves $k+1$ sets.

This completes the proof by strong induction. □

19. Prove $\sum_{k=1}^{n} 1/k^2 \leq 2 - 1/n$ for every n.

Proof. This clearly holds for $n = 1$. Assume it holds for some $n \geq 1$. Then $\sum_{k=1}^{n+1} 1/k^2 \leq 2 - 1/n + 1/(n+1)^2 = 2 - \frac{(n+1)^2 - n}{n(n+1)^2} \leq 2 - 1/(n+1)$. The proof is complete. □

21. If $n \in \mathbb{N}$, then $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} \geq 1 + \frac{n}{2}$.

Proof. If $n = 1$, the result is obvious.

Assume the proposition holds for some $n > 1$. Then

$$\begin{aligned}
&\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^{n+1}} \\
&= \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n}\right) + \left(\frac{1}{2^n+1} + \frac{1}{2^n+2} + \frac{1}{2^n+3} + \cdots + \frac{1}{2^{n+1}}\right) \\
&\geq \left(1 + \frac{n}{2}\right) + \left(\frac{1}{2^n+1} + \frac{1}{2^n+2} + \frac{1}{2^n+3} + \cdots + \frac{1}{2^{n+1}}\right).
\end{aligned}$$

Now, the sum $\left(\frac{1}{2^n+1} + \frac{1}{2^n+2} + \frac{1}{2^n+3} + \cdots + \frac{1}{2^{n+1}}\right)$ on the right has $2^{n+1} - 2^n = 2^n$ terms, all greater than or equal to $\frac{1}{2^{n+1}}$, so the sum is greater than $2^n \frac{1}{2^{n+1}} = \frac{1}{2}$. Therefore we get $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^{n+1}} \geq \left(1 + \frac{n}{2}\right) + \left(\frac{1}{2^n+1} + \frac{1}{2^n+2} + \frac{1}{2^n+3} + \cdots + \frac{1}{2^{n+1}}\right) \geq \left(1 + \frac{n}{2}\right) + \frac{1}{2} = 1 + \frac{n+1}{2}$. This means the result is true for $n+1$, so the theorem is proved. □

23. Use induction to prove the binomial theorem $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$.

Proof. For $n = 1$, this is $(x+y)^1 = \binom{1}{0}x^1y^0 + \binom{1}{1}x^0y^1 = x + y$, which is true.

Now assume the theorem is true for some $n > 1$. We will show that this implies that it is true for the power $n + 1$. Just observe that

$$\begin{aligned}
(x+y)^{n+1} &= (x+y)(x+y)^n \\
&= (x+y)\sum_{i=0}^{n}\binom{n}{i}x^{n-i}y^i \\
&= \sum_{i=0}^{n}\binom{n}{i}x^{(n+1)-i}y^i \quad + \quad \sum_{i=0}^{n}\binom{n}{i}x^{n-i}y^{i+1} \\
&= \sum_{i=0}^{n}\left[\binom{n}{i} + \binom{n}{i-1}\right]x^{(n+1)-i}y^i \qquad + y^{n+1} \\
&= \sum_{i=0}^{n}\binom{n+1}{i}x^{(n+1)-i}y^i \qquad + \binom{n+1}{n+1}y^{n+1} \\
&= \sum_{i=0}^{n+1}\binom{n+1}{i}x^{(n+1)-i}y^i.
\end{aligned}$$

This shows that the formula is true for $(x+y)^{n+1}$, so the theorem is proved. □

25. Concerning the Fibonacci sequence, prove that $F_1+F_2+F_3+F_4+\ldots+F_n = F_{n+2}-1$.

Proof. The proof is by induction.

(1) When $n = 1$ the statement is $F_1 = F_{1+2}-1 = F_3-1 = 2-1 = 1$, which is true. Also when $n = 2$ the statement is $F_1 + F_2 = F_{2+2} - 1 = F_4 - 1 = 3 - 1 = 2$, which is true, as $F_1 + F_2 = 1 + 1 = 2$.

(2) Now assume $k \geq 1$ and $F_1 + F_2 + F_3 + F_4 + \ldots + F_k = F_{k+2} - 1$. We need to show $F_1 + F_2 + F_3 + F_4 + \ldots + F_k + F_{k+1} = F_{k+3} - 1$. Observe that

$$\begin{aligned}
F_1 + F_2 + F_3 + F_4 + \ldots + F_k + F_{k+1} &= \\
(F_1 + F_2 + F_3 + F_4 + \ldots + F_k) + F_{k+1} &= \\
F_{k+2} - 1 + +F_{k+1} &= (F_{k+1} + F_{k+2}) - 1 \\
&= F_{k+3} - 1.
\end{aligned}$$

This completes the proof by induction. □

27. Concerning the Fibonacci sequence, prove that $F_1 + F_3 + \cdots + F_{2n-1} = F_{2n}$.

Proof. If $n = 1$, the result is immediate. Assume for some $n > 1$ we have $\sum_{i=1}^{n} F_{2i-1} = F_{2n}$. Then $\sum_{i=1}^{n+1} F_{2i-1} = F_{2n+1} + \sum_{i=1}^{n} F_{2i-1} = F_{2n+1} + F_{2n} = F_{2n+2} = F_{2(n+1)}$ as desired. □

29. Prove that $\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \binom{n-3}{3} + \cdots + \binom{1}{n-1} + \binom{0}{n} = F_{n+1}$.

Proof. (Strong Induction) For $n = 1$ this is $\binom{1}{0} + \binom{0}{1} = 1 + 0 = 1 = F_2 = F_{1+1}$. Thus the assertion is true when $n = 1$.

Now fix n and assume that $\binom{k}{0}+\binom{k-1}{1}+\binom{k-2}{2}+\binom{k-3}{3}+\cdots+\binom{1}{k-1}+\binom{0}{k}=F_{k+1}$ whenever $k<n$. In what follows we use the identity $\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-1}{k}$. We also often use $\binom{a}{b}=0$ whenever it is untrue that $0\leq b\leq a$.

$$\begin{aligned}
&\binom{n}{0}+\binom{n-1}{1}+\binom{n-2}{2}+\cdots+\binom{1}{n-1}+\binom{0}{n}\\
&=\binom{n}{0}+\binom{n-1}{1}+\binom{n-2}{2}+\cdots+\binom{1}{n-1}\\
&=\binom{n-1}{-1}+\binom{n-1}{0}+\binom{n-2}{0}+\binom{n-2}{1}+\binom{n-3}{1}+\binom{n-3}{2}+\cdots+\binom{0}{n-1}+\binom{0}{n}\\
&=\binom{n-1}{0}+\binom{n-2}{0}+\binom{n-2}{1}+\binom{n-3}{1}+\binom{n-3}{2}+\cdots+\binom{0}{n-1}+\binom{0}{n}\\
&=\left[\binom{n-1}{0}+\binom{n-2}{1}+\cdots+\binom{0}{n-1}\right]+\left[\binom{n-2}{0}+\binom{n-3}{1}+\cdots+\binom{0}{n-2}\right]\\
&=F_n+F_{n-1}=F_n
\end{aligned}$$

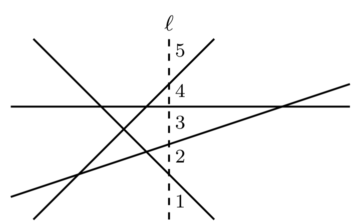
This completes the proof. □

31. Prove that $\sum_{k=0}^{n}\binom{k}{r}=\binom{n+1}{r+1}$, where $r\in\mathbb{N}$.
Hint: Use induction on the integer n. After doing the basis step, break up the expression $\binom{k}{r}$ as $\binom{k}{r}=\binom{k-1}{r-1}+\binom{k-1}{r}$. Then regroup, use the induction hypothesis, and recombine using the above identity.

33. Suppose that n infinitely long straight lines lie on the plane in such a way that no two are parallel, and no three intersect at a single point. Show that this arrangement divides the plane into $\frac{n^2+n+2}{2}$ regions.

Proof. The proof is by induction. For the basis step, suppose $n=1$. Then there is one line, and it clearly divides the plane into 2 regions, one on either side of the line. As $2=\frac{1^2+1+2}{2}=\frac{n^2+n+2}{2}$, the formula is correct when $n=1$.
Now suppose there are $n+1$ lines on the plane, and that the formula is correct for when there are n lines on the plane. Single out one of the $n+1$ lines on the plane, and call it ℓ. Remove line ℓ, so that there are now n lines on the plane.

By the induction hypothesis, these n lines divide the plane into $\frac{n^2+n+2}{2}$ regions. Now add line ℓ back. Doing this adds an additional $n+1$ regions. (The diagram illustrates the case where $n+1=5$. Without ℓ, there are $n=4$ lines. Adding ℓ back produces $n+1=5$ new regions.)



Thus, with $n+1$ lines there are all together $(n+1)+\frac{n^2+n+2}{2}$ regions. Observe

$$(n+1)+\frac{n^2+n+2}{2}=\frac{2n+2+n^2+n+2}{2}=\frac{(n+1)^2+(n+1)+2}{2}.$$

Thus, with $n+1$ lines, we have $\frac{(n+1)^2+(n+1)+2}{2}$ regions, which means that the formula is true for when there are $n+1$ lines. We have shown that if the formula is true for n lines, it is also true for $n+1$ lines. This completes the proof by induction. □

35. If $n, k \in \mathbb{N}$, and n is even and k is odd, then $\binom{n}{k}$ is even.

Proof. Notice that if k is not a value between 0 and n, then $\binom{n}{k} = 0$ is even; thus from here on we can assume that $0 < k < n$. We will use strong induction.

For the basis case, notice that the assertion is true for the even values $n = 2$ and $n = 4$: $\binom{2}{1} = 2$; $\binom{4}{1} = 4$; $\binom{4}{3} = 4$ (even in each case).
Now fix and even n assume that $\binom{m}{k}$ is even whenever m is even, k is odd, and $m < n$. Using the identity $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ three times, we get

$$\begin{aligned}\binom{n}{k} &= \binom{n-1}{k-1} + \binom{n-1}{k} \\ &= \binom{n-2}{k-2} + \binom{n-2}{k-1} + \binom{n-2}{k-1} + \binom{n-2}{k} \\ &= \binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k}.\end{aligned}$$

Now, $n-2$ is even, and k and $k-2$ are odd. By the inductive hypothesis, the outer terms of the above expression are even, and the middle is clearly even; thus we have expressed $\binom{n}{k}$ as the sum of three even integers, so it is even. □

37. Prove that if $m, n \in \mathbb{N}$, then $\sum_{k=0}^{n} k\binom{m+k}{m} = n\binom{m+n+1}{m+1} - \binom{m+n+1}{m+2}$.

Proof. We will use induction on n. Let m be any integer.

(1) If $n = 1$, then the equation is $\sum_{k=0}^{1} k\binom{m+k}{m} = 1\binom{m+1+1}{m+1} - \binom{m+1+1}{m+2}$, and this is $0\binom{m}{m} + 1\binom{m+1}{m} = 1\binom{m+2}{m+1} - \binom{m+2}{m+2}$, which yields the true statement $m+1 = m+2-1$.

(2) Now let $n > 1$ and assume the equation holds for n. (This is the inductive hypothesis.) Now we will confirm that it holds for $n+1$. Observe that

$$\sum_{k=0}^{n+1} k\binom{m+k}{m} = \qquad \text{(left-hand side for } n+1\text{)}$$

$$\sum_{k=0}^{n} k\binom{m+k}{m} + (n+1)\binom{m+(n+1)}{m} = \qquad \text{(split off final term)}$$

$$n\binom{m+n+1}{m+1} - \binom{m+n+1}{m+2} + (n+1)\binom{m+n+1}{m} = \qquad \text{(apply inductive hypothesis)}$$

$$n\binom{m+n+1}{m+1} + \binom{m+n+1}{m+1} - \binom{m+n+2}{m+2} + (n+1)\binom{m+n+1}{m} = \qquad \text{(Pascal's formula)}$$

$$(n+1)\binom{m+n+1}{m+1} - \binom{m+n+2}{m+2} + (n+1)\binom{m+n+1}{m} = \qquad \text{(factor)}$$

$$(n+1)\left[\binom{m+n+1}{m+1} + \binom{m+n+1}{m}\right] - \binom{m+n+2}{m+2} = \qquad \text{(factor again)}$$

$$(n+1)\binom{m+n+2}{m+1} - \binom{m+n+2}{m+2} = \qquad \text{(Pascal's formula)}$$

$$(n+1)\binom{m+(n+1)+1}{m+1} - \binom{m+(n+1)+1}{m+2}. \qquad \text{(right-hand side for } n+1\text{)}$$

The proof is done. □

39. If n and k are non-negative integers, then $\binom{n+0}{0}+\binom{n+1}{1}+\cdots+\binom{n+k}{k}=\binom{n+k+1}{k}$.

Proof. We will use induction on k. Let n be any non-negative integer.

(1) If $k=0$, then the equation is $\binom{n+0}{0}=\binom{n+0+1}{0}$, which reduces to $1=1$.

(2) Assume the equation holds for some $k\geq 1$. (This is the inductive hypothesis.) Now we will show that it holds for $k+1$. The left side for $k+1$ is

$$\begin{aligned}
&\binom{n+0}{0}+\binom{n+1}{1}+\cdots+\binom{n+k}{k}+\binom{n+(k+1)}{k+1}\\
&=\binom{n+k+1}{k}+\binom{n+k+1}{k+1} && \text{(apply inductive hypothesis)}\\
&=\binom{n+k+2}{k+1} && \text{(Pascal's formula)}\\
&=\binom{n+(k+1)+1}{k+1}. && \text{(right-hand side for } k+1\text{)}
\end{aligned}$$

The proof is complete. □

41. Prove that $\sum\limits_{k=0}^{m}\binom{m}{k}\binom{n}{p+k}=\binom{m+n}{m+p}$ for non-negative integers m, n and p.

Proof. We will use induction on n. Let m and p be any non-negative integers.

(1) If $n=0$, then the equation is $\sum\limits_{k=0}^{m}\binom{m}{k}\binom{0}{p+k}=\binom{m+0}{m+p}$. This holds if $p>0$, because then $\binom{0}{p+k}=0=\binom{m}{m+p}$, and both sides of the equation are zero. If $p=0$, the equation is $\sum\limits_{k=0}^{m}\binom{m}{k}\binom{0}{k}=\binom{m}{m}$, and both sides equal 1.

(2) Now take $n\geq 1$ and suppose the equation holds for n. (This is the inductive hypothesis.) Next we confirm that the equation holds for $n+1$.

$$\begin{aligned}
&\binom{m+(n+1)}{m+p} && \text{(right-hand side for } n+1\text{)}\\
&=\binom{m+n}{m+(p-1)}+\binom{m+n}{m+p} && \text{(Pascal's formula)}\\
&=\sum_{k=0}^{m}\binom{m}{k}\binom{n}{(p-1)+k}+\sum_{k=0}^{m}\binom{m}{k}\binom{n}{p+k} && \text{(apply inductive hypothesis)}\\
&=\sum_{k=0}^{m}\binom{m}{k}\left[\binom{n}{(p-1)+k}+\binom{n}{p+k}\right] && \text{(combine)}\\
&=\sum_{k=0}^{m}\binom{m}{k}\binom{n+1}{p+k} && \text{(Pascal's formula)}
\end{aligned}$$

This final expression is left-hand side for $n+1$, so the proof is finished. □