Probabilistic proofs of hook length formulas involving trees

by

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Let $T$ be a rooted tree with $n$ distinguishable vertices. We use $T$ to stand for the vertex set of $T$. An increasing labeling of $T$ is a bijection $\ell : T \to \{1, 2, \ldots, n\}$ such that $\ell(v) \leq \ell(w)$ for all descendents $w$ of $v$. Let $f_T$ be the number of increasing labelings. The hook length, $h_v$, of a vertex $v$ is the number of descendents of $v$ (including $v$ itself). The hook length formula for trees states that

$$f_T = \frac{n!}{\prod_{v \in T} h_v}.$$ 

There is a similar formula for the number of standard Young tableaux of given shape where a hook length is the cardinality of a set which resembles a physical hook. Greene, Nijenhuis, and Wilf gave a beautiful probabilistic proof of the tableau formula where the hooklengths enter in a very natural way.

Recently, Han discovered a formula which has the interesting property that hooklengths appear as exponents. Specifically, let $B(n)$ be the set of all $n$-vertex binary trees (each vertex has no children, a left child, a right child, or both children). Han proved that

$$\sum_{T \in B(n)} \prod_{v \in T} \frac{1}{h_v 2^{h_v - 1}} = \frac{1}{n!}$$

using algebraic manipulations. We will show how to give a simple probabilistic proof of this equation as well as various generalizations. We will also pose some open questions raised by this work.