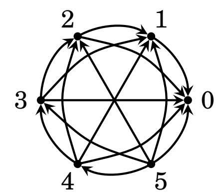
16.7 Solutions for Chapter 16

Section 16.1 Exercises

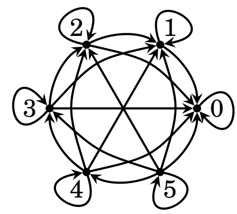
1. Let $A = \{0,1,2,3,4,5\}$. Write out the relation R that expresses $>$ on A. Then illustrate it with a diagram.

$$R = \{(5,4),(5,3),(5,2),(5,1),(5,0),(4,3),(4,2),(4,1),(4,0),(3,2),(3,1),(3,0),(2,1),(2,0),(1,0)\}$$



3. Let $A = \{0,1,2,3,4,5\}$. Write out the relation R that expresses $\geq$ on A. Then illustrate it with a diagram.

$$\begin{aligned} R \;=\; \{&(5,5),(5,4),(5,3),(5,2),(5,1),(5,0),\\ &(4,4),(4,3),(4,2),(4,1),(4,0),\\ &(3,3),(3,2),(3,1),(3,0),\\ &(2,2),(2,1),(2,0),(1,1),(1,0),(0,0)\} \end{aligned}$$



5. The following diagram represents a relation R on a set A. Write the sets A and R. Answer: $A = \{0,1,2,3,4,5\}$; $R = \{(3,3),(4,3),(4,2),(1,2),(2,5),(5,0)\}$

7. Write the relation $<$ on the set $A = \mathbb{Z}$ as a subset R of $\mathbb{Z} \times \mathbb{Z}$. This is an infinite set, so you will have to use set-builder notation.
Answer: $R = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} : y - x \in \mathbb{N}\}$

9. How many different relations are there on the set $A = \{1,2,3,4,5,6\}$?
Consider forming a relation $R \subseteq A \times A$ on A. For each ordered pair $(x,y) \in A \times A$, we have two choices: we can either include (x,y) in R or not include it. There are $6 \cdot 6 = 36$ ordered pairs in $A \times A$. By the multiplication principle, there are thus 2^{36} different subsets R and hence also this many relations on A.

11. Answer: $2^{(|A|^2)}$ **13.** Answer: $\neq$ **15.** Answer: $\equiv \pmod 3$

Section 16.2 Exercises

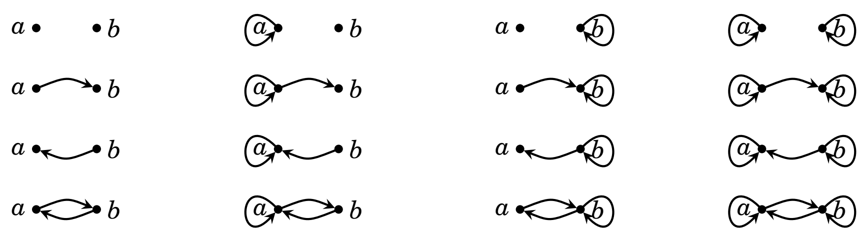
1. Consider the relation $R = \{(a,a),(b,b),(c,c),(d,d),(a,b),(b,a)\}$ on the set $A = \{a,b,c,d\}$. Which of the properties reflexive, symmetric and transitive does R possess and why? If a property does not hold, say why.
This **is reflexive** because $(x,x) \in R$ (i.e., xRx)for every $x \in A$.
It **is symmetric** because it is impossible to find an $(x,y) \in R$ for which $(y,x) \notin R$.
It **is transitive** because $(xRy \wedge yRz) \Rightarrow xRz$ always holds.

3. Consider the relation $R = \{(a,b),(a,c),(c,b),(b,c)\}$ on the set $A = \{a,b,c\}$. Which of the properties reflexive, symmetric and transitive does R possess and why? If a property does not hold, say why.

This **is not reflexive** because $(a,a) \notin R$ (for example).
It **is not symmetric** because $(a,b) \in R$ but $(b,a) \notin R$.
It **is not transitive** because cRb and bRc are true, but cRc is false.

5. Consider the relation $R = \{(0,0),(\sqrt{2},0),(0,\sqrt{2}),(\sqrt{2},\sqrt{2})\}$ on $\mathbb{R}$. Say whether this relation is reflexive, symmetric and transitive. If a property does not hold, say why.
This **is not reflexive** because $(1,1) \notin R$ (for example).
It **is symmetric** because it is impossible to find an $(x,y) \in R$ for which $(y,x) \notin R$.
It **is transitive** because $(xRy \wedge yRz) \Rightarrow xRz$ always holds.

7. There are 16 possible different relations R on the set $A = \{a,b\}$. Describe all of them. (A picture for each one will suffice, but don't forget to label the nodes.) Which ones are reflexive? Symmetric? Transitive?



Only the four in the right column are reflexive. Only the eight in the first and fourth rows are symmetric. All of them are transitive **except** the first three on the fourth row.

9. Define a relation on $\mathbb{Z}$ by declaring xRy if and only if x and y have the same parity. Say whether this relation is reflexive, symmetric and transitive. If a property does not hold, say why. What familiar relation is this?
This **is reflexive** because xRx since x always has the same parity as x.
It **is symmetric** because if x and y have the same parity, then y and x must have the same parity (that is, $xRy \Rightarrow yRx$).
It **is transitive** because if x and y have the same parity and y and z have the same parity, then x and z must have the same parity. (That is $(xRy \wedge yRz) \Rightarrow xRz$ always holds.)
The relation is congruence modulo 2.

11. Suppose $A = \{a,b,c,d\}$ and $R = \{(a,a),(b,b),(c,c),(d,d)\}$. Say whether this relation is reflexive, symmetric and transitive. If a property does not hold, say why.
This **is reflexive** because $(x,x) \in R$ for every $x \in A$.
It **is symmetric** because it is impossible to find an $(x,y) \in R$ for which $(y,x) \notin R$.
It **is transitive** because $(xRy \wedge yRz) \Rightarrow xRz$ always holds.
(For example $(aRa \wedge aRa) \Rightarrow aRa$ is true, etc.)

13. Consider the relation $R = \{(x,y) \in \mathbb{R} \times \mathbb{R} : x - y \in \mathbb{Z}\}$ on $\mathbb{R}$. Prove that this relation is reflexive and symmetric, and transitive.

Proof. In this relation, xRy means $x-y \in \mathbb{Z}$.

To see that R is reflexive, take any $x \in \mathbb{R}$ and observe that $x-x=0 \in \mathbb{Z}$, so xRx. Therefore R is reflexive.

To see that R is symmetric, we need to prove $xRy \Rightarrow yRx$ for all $x,y \in \mathbb{R}$. We use direct proof. Suppose xRy. This means $x-y \in \mathbb{Z}$. Then it follows that $-(x-y)=y-x$ is also in $\mathbb{Z}$. But $y-x \in \mathbb{Z}$ means yRx. We've shown xRy implies yRx, so R is symmetric.

To see that R is transitive, we need to prove $(xRy \wedge yRz) \Rightarrow xRz$ is always true. We prove this conditional statement with direct proof. Suppose xRy and yRz. Since xRy, we know $x-y \in \mathbb{Z}$. Since yRz, we know $y-z \in \mathbb{Z}$. Thus $x-y$ and $y-z$ are both integers; by adding these integers we get another integer $(x-y)+(y-z)=x-z$. Thus $x-z \in \mathbb{Z}$, and this means xRz. We've now shown that if xRy and yRz, then xRz. Therefore R is transitive. ■

15. Prove or disprove: If a relation is symmetric and transitive, then it is also reflexive.
This is **false**. For a counterexample, consider the relation $R=\{(a,a),(a,b),(b,a),(b,b)\}$ on the set $A=\{a,b,c\}$. This is symmetric and transitive but it is not reflexive.

17. Define a relation $\sim$ on $\mathbb{Z}$ as $x \sim y$ if and only if $|x-y| \le 1$. Say whether $\sim$ is reflexive, symmetric and transitive.
This is reflexive because $|x-x|=0 \le 1$ for all integers x. It is symmetric because $x \sim y$ if and only if $|x-y| \le 1$, if and only if $|y-x| \le 1$, if and only if $y \sim x$. It is not transitive because, for example, $0 \sim 1$ and $1 \sim 2$, but is not the case that $0 \sim 2$.

Section 16.3 Exercises

1. Let $A=\{1,2,3,4,5,6\}$, and consider the following equivalence relation on A: $R=\{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(2,3),(3,2),(4,5),(5,4),(4,6),(6,4),(5,6),(6,5)\}$. List the equivalence classes of R.

The equivalence classes are: $[1]=\{1\}$; $[2]=[3]=\{2,3\}$; $[4]=[5]=[6]=\{4,5,6\}$.

3. Let $A=\{a,b,c,d,e\}$. Suppose R is an equivalence relation on A. Suppose R has three equivalence classes. Also aRd and bRc. Write out R as a set.
Answer: $R=\{(a,a),(b,b),(c,c),(d,d),(e,e),(a,d),(d,a),(b,c),(c,b)\}$.

5. There are two different equivalence relations on the set $A=\{a,b\}$. Describe them all. Diagrams will suffice.
Answer: $R=\{(a,a),(b,b)\}$ and $R=\{(a,a),(b,b),(a,b),(b,a)\}$

7. Define a relation R on $\mathbb{Z}$ as xRy if and only if $3x-5y$ is even. Prove R is an equivalence relation. Describe its equivalence classes.

To prove that R is an equivalence relation, we must show it's reflexive, symmetric and transitive.
The relation R is reflexive for the following reason. If $x \in \mathbb{Z}$, then $3x-5x=-2x$ is even. But then since $3x-5x$ is even, we have xRx. Thus R is reflexive.

To see that R is symmetric, suppose xRy. We must show yRx. Since xRy, we know $3x-5y$ is even, so $3x-5y=2a$ for some integer a. Now reason as follows:

$$\begin{aligned}
3x-5y &= 2a \\
3x-5y+8y-8x &= 2a+8y-8x \\
3y-5x &= 2(a+4y-4x).
\end{aligned}$$

From this it follows that $3y-5x$ is even, so yRx. We've now shown xRy implies yRx, so R is symmetric.

To prove that R is transitive, assume that xRy and yRz. (We will show that this implies xRz.) Since xRy and yRz, it follows that $3x-5y$ and $3y-5z$ are both even, so $3x-5y=2a$ and $3y-5z=2b$ for some integers a and b. Adding these equations, we get $(3x-5y)+(3y-5z)=2a+2b$, and this simplifies to $3x-5z=2(a+b+y)$. Therefore $3x-5z$ is even, so xRz. We've now shown that if xRy and yRz, then xRz, so R is transitive.

We've now shown that R is reflexive, symmetric and transitive, so it is an equivalence relation.

The completes the first part of the problem. Now we move on the second part. To find the equivalence classes, first note that

$$[0]=\{x\in\mathbb{Z}:xR0\}=\{x\in\mathbb{Z}:3x-5\cdot 0 \text{ is even}\}=\{x\in\mathbb{Z}:3x \text{ is even}\}=\{x\in\mathbb{Z}:x \text{ is even}\}.$$

Thus the equivalence class [0] consists of all even integers. Next, note that

$$[1]=\{x\in\mathbb{Z}:xR1\}=\{x\in\mathbb{Z}:3x-5\cdot 1 \text{ is even}\}=\{x\in\mathbb{Z}:3x-5 \text{ is even}\}=\big\{x\in\mathbb{Z}:x \text{ is odd}\big\}.$$

Thus the equivalence class [1] consists of all odd integers.

Consequently there are just two equivalence classes $\{\ldots,-4,-2,0,2,4,\ldots\}$ and $\{\ldots,-3,-1,1,3,5,\ldots\}$.

9. Define a relation R on $\mathbb{Z}$ as xRy if and only if $4\mid(x+3y)$. Prove R is an equivalence relation. Describe its equivalence classes.

This is reflexive, because for any $x\in\mathbb{Z}$ we have $4\mid(x+3x)$, so xRx.

To prove that R is symmetric, suppose xRy. Then $4\mid(x+3y)$, so $x+3y=4a$ for some integer a. Multiplying by 3, we get $3x+9y=12a$, which becomes $y+3x=12a-8y$. Then $y+3x=4(3a-2y)$, so $4\mid(y+3x)$, hence yRx. Thus we've shown xRy implies yRx, so R is symmetric.

To prove transitivity, suppose xRy and yRz. Then $4|(x+3y)$ and $4|(y+3z)$, so $x+3y=4a$ and $y+3z=4b$ for some integers a and b. Adding these two equations produces $x+4y+3z=4a+4b$, or $x+3z=4a+4b-4y=4(a+b-y)$. Consequently $4|(x+3z)$, so xRz, and R is transitive.

As R is reflexive, symmetric and transitive, it is an equivalence relation.

Now let's compute its equivalence classes.

$[0]=\{x\in\mathbb{Z}:xR0\}=\{x\in\mathbb{Z}:4\mid(x+3\cdot 0)\}=\{x\in\mathbb{Z}:4\mid x\}= \quad \{\ldots-4,0,4,8,12,16\ldots\}$

$[1] = \{x \in \mathbb{Z} : xR1\} = \{x \in \mathbb{Z} : 4 \mid (x+3\cdot 1)\} = \{x \in \mathbb{Z} : 4 \mid (x+3)\} = \{\ldots -3, 1, 5, 9, 13, 17\ldots\}$
$[2] = \{x \in \mathbb{Z} : xR2\} = \{x \in \mathbb{Z} : 4 \mid (x+3\cdot 2)\} = \{x \in \mathbb{Z} : 4 \mid (x+6)\} = \{\ldots -2, 2, 6, 10, 14, 18\ldots\}$
$[3] = \{x \in \mathbb{Z} : xR3\} = \{x \in \mathbb{Z} : 4 \mid (x+3\cdot 3)\} = \{x \in \mathbb{Z} : 4 \mid (x+9)\} = \{\ldots -1, 3, 7, 11, 15, 19\ldots\}$

11. Prove or disprove: If R is an equivalence relation on an infinite set A, then R has infinitely many equivalence classes.
This is **False**. Counterexample: consider the relation of congruence modulo 2. It is a relation on the infinite set $\mathbb{Z}$, but it has only two equivalence classes.

13. Answer: $m|A|$

15. Answer: 15

Section 16.4 Exercises

1. List all the partitions of the set $A = \{a, b\}$. Compare your answer to the answer to Exercise 5 of Section 16.3.
There are just two partitions $\{\{a\},\{b\}\}$ and $\{\{a,b\}\}$. These correspond to the two equivalence relations $R_1 = \{(a,a),(b,b)\}$ and $R_2 = \{(a,a),(a,b),(b,a),(b,b)\}$, respectively, on A.

3. Describe the partition of $\mathbb{Z}$ resulting from the equivalence relation $\equiv \pmod 4$.
Answer: The partition is $\{[0],[1],[2],[3]\} =$
$\big\{\{\ldots,-4,0,4,8,12,\ldots\},\{\ldots,-3,1,5,9,13,\ldots\},\{\ldots,-2,2,4,6,10,14,\ldots\},\{\ldots,-1,3,7,11,15,\ldots\}\big\}$

5. Answer: Congruence modulo 2, or "same parity."

Section 16.5 Exercises

1. Write the addition and multiplication tables for $\mathbb{Z}_2$.

+	[0]	[1]
[0]	[0]	[1]
[1]	[1]	[0]

·	[0]	[1]
[0]	[0]	[0]
[1]	[0]	[1]

3. Write the addition and multiplication tables for $\mathbb{Z}_4$.

+	[0]	[1]	[2]	[3]
[0]	[0]	[1]	[2]	[3]
[1]	[1]	[2]	[3]	[0]
[2]	[2]	[3]	[0]	[1]
[3]	[3]	[0]	[1]	[2]

·	[0]	[1]	[2]	[3]
[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]
[2]	[0]	[2]	[0]	[2]
[3]	[0]	[3]	[2]	[1]

5. Suppose $[a],[b] \in \mathbb{Z}_5$ and $[a]\cdot[b] = [0]$. Is it necessarily true that either $[a] = [0]$ or $[b] = [0]$?

The multiplication table for $\mathbb{Z}_5$ is shown in Section 16.5. In the body of that table, the only place that [0] occurs is in the first row or the first column. That row and column are both headed by [0]. It follows that if $[a]\cdot[b]=[0]$, then either $[a]$ or $[b]$ must be [0].

7. Do the following calculations in $\mathbb{Z}_9$, in each case expressing your answer as $[a]$ with $0 \le a \le 8$.
(a) $[8]+[8]=[7]$ **(b)** $[24]+[11]=[8]$ **(c)** $[21]\cdot[15]=[0]$ **(d)** $[8]\cdot[8]=[1]$